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Extensions of RT_0 Topological Spaces of Fuzzy Sets

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Abstract

The aim of this paper is to study extensions of RT_0 topological spaces of fuzzy sets. We also construct RT_0 principal extensions of RT_0 topological spaces of fuzzy sets with the α-graded trace system for each α in $(0, 1]$.

Keywords: *Fuzzy topology, principal extension, remoted neighbourhood, RT_0 space, α-graded trace system.*

1 Introduction

In crisp topology, extension theory is a well developed theory (for references please see [2], [3],[9], [12], [19] and [20]). In fuzzy topology only some particular type of extensions such as compactifications, completions of fuzzy topological spaces and fuzzy uniform spaces have been studied in [15], [23], [24]. The fuzzyfication of general extension theory has been started by us in [5], where a concept of fuzzyfication of extensions of topological spaces of fuzzy sets is introduced and a method of construction of strongly T_0 principal extension of a strongly T_0 topological space of fuzzy sets is provided.

In this paper we study extension theory and provide a method of construction of RT_0 principal extension of an RT_0 topological space of fuzzy sets with the given α-graded trace system for each $\alpha \in (0, 1]$. In this setting for each $\alpha \in (0, 1]$, we find an RT_0 principal extension of an RT_0 topological space

(X, u) with the given α-graded trace system.
Chang [4] introduced the notion of fuzzy topological spaces. In this context it is worth noting that Chang's fuzzy topology is in fact a crisp topology of fuzzy sets. In this paper Chang's fuzzy topology will be referred to as topology of fuzzy sets. (X, u) will be called a topological space of fuzzy sets if X is a set and u is a Chang topology on it.
In Section 2, some known definitions and known results are given which will be used in the sequel.
In Section 3, a definition of RT_0 topological spaces of fuzzy sets is given. Some results concerning principal extensions have been established.
In Section 4, using the concepts and results of Section 3, we present a construction of RT_0 principal extension of RT_0 spaces with the given α-graded trace system.

2 Preliminaries

Let X be a nonempty set and Y be a nonempty subset of X. For a fuzzy set λ of Y, its *natural extension* $\lambda_{Y<X}$ is defined by $\lambda_{Y<X}(x) = \lambda(x)$ if $x \in Y$ and $\lambda_{Y<X}(x) = 0$ if $x \in X - Y$. When there is no chance of confusion, we shall use (for simplicity) the same symbol λ for $\lambda_{Y<X}$.
In what follows I will stand for [0,1].

Definition 2.1 *[14] Let (X, u) be a topological space of fuzzy sets. Then (X, u) is called T_0 if for any pair of distinct points $x, y \in X, \exists \lambda \in u$ such that $\lambda(x) \neq \lambda(y)$.*

Definition 2.2 *[4] Let (X, u) and (Y, v) be two topological spaces of fuzzy sets. A mapping $\eta : (X, u) \to (Y, v)$ is said to be continuous if $\eta^{-1}(\lambda) \in u, \forall \lambda \in v$.*

Definition 2.3 *[5] Let (X, u) and (Y, v) be two topological spaces of fuzzy sets. A mapping $\eta : (X, u) \to (Y, v)$ is said to be closed if $\eta(\mu) \in v', \forall \mu \in u'$, where u' and v' are the families of closed sets in (X, u) and (Y, v) respectively.*

Definition 2.4 *[11] Let (X, u) and (Y, v) be two topological spaces of fuzzy sets. A mapping $\eta : (X, u) \to (Y, v)$ is said to be open if $\eta(\lambda) \in v, \forall \lambda \in u$.*

Definition 2.5 *[11] A mapping $\eta : (X, u) \to (Y, v)$ is said to be a homeomorphism if η is bijective, continuous and open (or closed).*

Definition 2.6 *[25] Let (X, u) be a topological space of fuzzy sets and λ be a fuzzy set in X. Then the closure of λ in (X, u) is defined by*

$$cl_u\lambda = \wedge\{\mu \in u' : \mu \geq \lambda\}.$$

When there is no chance of confusion regarding the role of u, $cl_u\lambda$ will simply be denoted by $cl\lambda$.

Theorem 2.7 *[16, 17] Let (X,u) and (Y,v) be two topological spaces of fuzzy sets and $\eta : X \to Y$ be a mapping. Then $\eta : (X,u) \to (Y,v)$ is continuous if and only if*

$$\eta(cl_u\lambda) \leq cl_v\eta(\lambda), \forall\lambda \in I^X.$$

Theorem 2.8 *[5] For a bijective mapping $\eta : X \to Y, \eta : (X,u) \to (Y,v)$ is homeomorphism if and only if*

$$\eta(cl_u\lambda) = cl_v\eta(\lambda),\ \forall\lambda \in I^X.$$

Definition 2.9 *[25] Let (X,u) be a topological space of fuzzy sets and $A \subset X$. Let λ be a fuzzy set in X. Then λ_A is a fuzzy set in A defined by*

$$\lambda_A(x) = \lambda(x), \forall x \in A.$$

Define $u_A = \{\lambda_A : \lambda \in u\}$. Then it is easily verified that u_A is a topology of fuzzy sets on A and (A, u_A) is called a subspace of (X,u).

Definition 2.10 *[1] A fuzzy stack S on X is a subset of I^X such that $\lambda \geq \mu \in S$ implies $\lambda \in S$.*

Definition 2.11 *[1] A fuzzy grill G on X is a fuzzy stack on X such that*
(i) $\tilde{0}_X \notin G$,
(ii) $\lambda \vee \mu \in G \Rightarrow \lambda \in G$ or $\mu \in G$.

Remark 2.12 *In this article fuzzy stacks and fuzzy grills as defined in [definitions 2.10 and 2.11] will be referred to as stacks of fuzzy sets and grills of fuzzy sets respectively.*

A grill of fuzzy sets G is called proper if $G \neq \phi$.

Definition 2.13 *[5] A grill G of fuzzy sets on a topological space (X,u) is said to be a c-grill of fuzzy sets if $cl\lambda \in G \Rightarrow \lambda \in G, \quad \forall\lambda \in I^X$.*

Definition 2.14 $\forall f \in I^X$*, we define $Z(f)$ to be the subset $\{x \in X : f(x) = 0\}$ of X which is called the zero-set of f in X.*

Remark 2.15 *Here it is important to note that the symbol $Z(f)$ has been used by Gillman and Jerison [13] for the zero-set of a real valued continuous function f on a topological space X. In this article we use the same symbol for the zero-set of an arbitrary element $f \in I^X$ for an arbitrary set X.*

Definition 2.16 *[5] Let (X,u) and (Y,v) be two topological spaces of fuzzy sets and $\eta : X \to Y$ be a mapping. Then $(\eta, (Y,v))$ is said to be an embedding of (X,u) if $\eta : (X,u) \to (\eta(X), v_{\eta(X)})$ is a homeomorphism.*

Definition 2.17 *[5] Let (X,u) and (Y,v) be two topological spaces of fuzzy sets and $\eta : X \to Y$ be a mapping. Then $(\eta,(Y,v))$ is said to be an extension of (X,u) if $(\eta,(Y,v))$ is an embedding and $cl_v\eta(\tilde{1}_X) = \tilde{1}_Y$ or equivalently $cl_v\tilde{1}_{\eta(X)} = \tilde{1}_Y$, subject to the assumption that $\tilde{1}_{\eta(X)}$ is the fuzzy set in Y satisfying $\tilde{1}_{\eta(X)}(y) = 1, \forall y \in \eta(X)$ and $\tilde{1}_{\eta(X)}(y) = 0, \forall y \in Y - \eta(X)$.*

Theorem 2.18 *[5] If $\eta : X \to Y$ is one-one and (X,u) , (Y,v) are topological spaces of fuzzy sets, then $(\eta,(Y,v))$ is an extension of (X,u) if and only if*
(i) $\forall \lambda \in I^X, \eta(cl_u\lambda) = (cl_v\eta(\lambda)) \wedge \eta(\tilde{1}_X)$,
and
(ii) $cl_v\eta(\tilde{1}_X) = \tilde{1}_Y$.

Definition 2.19 *[5] Let $E_1 = (\eta_1,(Y_1,v_1))$ and $E_2 = (\eta_2,(Y_2,v_2))$ be two extensions of (X,u). Then E_1 is said to be greater than or equal to E_2 (written as $E_1 \geq E_2$) if there is a continuous function f from (Y_1,v_1) onto (Y_2,v_2) such that $fo\eta_1 = \eta_2$.*

Definition 2.20 *[5] The extension $E_1 = (\eta_1,(Y_1,v_1))$ is said to be equivalent to the extension $E_2 = (\eta_2,(Y_2,v_2))$ (written as $E_1 \approx E_2$) if there is a homeomorphism h of (Y_1,v_1) onto (Y_2,v_2) such that $ho\eta_1 = \eta_2$.*

Definition 2.21 *[5] Let (X,u) be a topological space of fuzzy sets and $\mathcal{B}$ be a family of closed sets in (X,u). Then $\mathcal{B}$ is said to be a base for the closed sets in (X,u) if each closed set in (X,u) can be expressed as the infimum of a subfamily of $\mathcal{B}$.*

Theorem 2.22 *[5] Let $\mathcal{B} \subset I^X$ such that*
(i) $\tilde{0}_X \in \mathcal{B}$,
(ii) $\forall \lambda_1, \lambda_2 \in \mathcal{B} \Rightarrow \lambda_1 \vee \lambda_2 \in \mathcal{B}$.
Then $\mathcal{B}$ is a base for closed sets of some topology of fuzzy sets on X.

Definition 2.23 *[5] An extension $E = (\eta,(Y,v))$ is said to be a principal extension of (X,u) if $\{cl_v\eta(\mu) : \mu \in I^X\}$ is a base for the closed sets in (Y,v).*

Definition 2.24 *[18] A fuzzy point in a set X is a mapping $\alpha_x : X \to I$, where $x \in X, \alpha \in (0,1]$ defined by $\alpha_x(x) = \alpha$ and $\alpha_x(y) = 0$ for $y \neq x$. Here x is the support of the fuzzy point α_x and α its value.*
A fuzzy point α_x is said to belong to a fuzzy set λ in X, denoted by $\alpha_x \tilde{\in} \lambda$ if $\alpha \leq \lambda(x)$.

Following [21] a definition of *remoted neighbourhood* of a fuzzy point is given below:

Definition 2.25 *Let (X, u) be a topological space of fuzzy sets and α_x be a fuzzy point. Then $\lambda \in u'$ is called a remoted neighbourhood of α_x if $\alpha_x \not\tilde{\in} \lambda$. The set of all remoted neighbourhoods of α_x is denoted by R_{α_x}.*

Definition 2.26 *[5] A topological space (X, u) of fuzzy sets is called strongly T_0 if for each pair of distinct points $x, y \in X$, either there is a $\lambda \in u$ such that $\lambda(x) > 0$ and $\lambda(y) = 0$ or there is a $\mu \in u$ such that $\mu(x) = 0$ and $\mu(y) > 0$.*

3 Some Basic Results on Extensions of Topological Spaces of Fuzzy Sets

We begin the section with the following definition.

Definition 3.1 *A topological space (X, u) of fuzzy sets is said to be RT_0 if for each pair of distinct points x, y of X and for each $\alpha \in (0, 1]$, $\exists \lambda_\alpha \in R_{\alpha_x}, \lambda_\alpha \notin R_{\alpha_y}$ or $\exists \mu_\alpha \in R_{\alpha_y}$, $\mu_\alpha \notin R_{\alpha_x}$.*

Example 3.2 *Let $X = \{x, y\}$ and $u = \Big\{0_X, 1_X\Big\} \bigcup \Big\{ \ \{x/\alpha, y/1\} : \alpha \in [0, 1) \ \Big\}$. Then $u' = \Big\{1_X, 0_X\Big\} \bigcup \Big\{ \ \{x/\alpha, y/0\} : \alpha \in (0, 1] \ \Big\}$.
Thus for each $\alpha \in (0, 1]$, $\exists \ \lambda_\alpha = \{x/\alpha, y/0\} \in u'$ such that $\alpha > 0 = \lambda_\alpha(y)$ and $\alpha \leq \alpha = \lambda_\alpha(x)$. i.e., $\alpha_y \not\tilde{\in} \lambda_\alpha$ and $\alpha_x \tilde{\in} \lambda_\alpha$. i.e., $\lambda_\alpha \in R_{\alpha_y}$ and $\lambda_\alpha \notin R_{\alpha_x}$.
Therefore (X, u) is an RT_0-topological space of fuzzy sets.*

Theorem 3.3 *If (X, u) is RT_0, then it is strongly T_0.*

Proof. Let (X, u) be RT_0 and $x, y \in X$ such that $x \neq y$. Then for each $\alpha \in (0, 1]$, $\exists \lambda_\alpha \in R_{\alpha_x}, \lambda_\alpha \notin R_{\alpha_y}$ or $\exists \mu_\alpha \in R_{\alpha_y}, \mu_\alpha \notin R_{\alpha_x}$.
Therefore for each $\alpha \in (0, 1]$, $\exists \ \lambda_\alpha \in u'$ such that $\alpha > \lambda_\alpha(x), \alpha \leq \lambda_\alpha(y)$ or $\exists \ \mu_\alpha \in u'$ such that $\alpha > \mu_\alpha(y), \alpha \leq \mu_\alpha(x)$.
Thus for $\alpha = 1, \exists \ \lambda_1 \in u'$ such that $\lambda_1(x) < 1, \lambda_1(y) = 1$ or $\exists \ \mu_1 \in u'$ such that $\mu_1(y) < 1, \mu_1(x) = 1$.
Taking $\lambda_1' = \gamma$ and $\mu_1' = \delta$ we have $\exists \ \gamma \in u$ such that $\gamma(x) > 0, \gamma(y) = 0$ or $\exists \ \delta \in u$ such that $\delta(y) > 0, \delta(x) = 0$.
Hence (X, u) is strongly T_0.

Note 3.4 *But the converse of Theorem 3.3 is not true, which is justified by the following Example.*

Example 3.5 *Let $X = \{x, y\}$, $u = \Big\{ \ \tilde{0}_X, \tilde{1}_X, \{x/0.4, y/0\} \ \Big\}$.
Then $u' = \Big\{ \ \tilde{1}_X, \tilde{0}_X, \{x/0.6, y/1\} \ \Big\}$.
If $\alpha = 0.5$, then $R_{\alpha_x} = R_{\alpha_y}$. Thus (X, u) is not RT_0.
But it is clear that (X, u) is strongly T_0.*

Theorem 3.6 *If (X,u) is RT_0, then it is T_0.*

Proof. Let (X,u) be RT_0. Then it is strongly T_0 and hence it is T_0.

Note 3.7 *But the converse of the theorem is not true, which is justified by the following example.*

Example 3.8 *Let* $X=\{x,y,z\},\ u=\left\{\ \tilde{0}_X,\tilde{1}_X,\{x/0.2,y/0.3,z/0.4\}\ \right\}$.
Therefore $u'=\left\{\ \tilde{1}_X,\tilde{0}_X,\{x/0.8,y/0.7,z/0.6\}\ \right\}$.
If $\alpha=0.5$, *then* $R_{\alpha_x}=R_{\alpha_y}=R_{\alpha_z}$. *Therefore* (X,u) *is not* RT_0.
It is easy to check that (X,u) *is* T_0.

Definition 3.9 *Let (X,u) be a topological space of fuzzy sets.* $\forall x\in X,\ \ \forall\alpha\in(0,1]$, *define*

$$G_{\alpha_x}=\{\ \lambda\in I^X:\alpha_x\tilde{\in}cl\lambda\ \}.$$

Theorem 3.10 *Let (X,u) be a topological space of fuzzy sets. Then (X,u) is RT_0 if and only if $\forall x,y\in X, G_{\alpha_x}=G_{\alpha_y}$ for some $\alpha\in(0,1]$ imply $x=y$.*

Proof. Let (X,u) be an RT_0 topological space of fuzzy sets. Let $\alpha\in(0,1]$ and $x,y\in X$ such that $x\neq y$.
Since(X,u) is RT_0,

$$\exists\lambda\in R_{\alpha_x},\ \lambda\notin R_{\alpha_y}, \tag{1}$$

or

$$\exists\mu\in R_{\alpha_y},\ \mu\notin R_{\alpha_x}. \tag{2}$$

Without any loss of generality we assume that (1) holds.
Then $\alpha_x\ \tilde{\not\in}\lambda=cl\lambda,\ \ \alpha_y\tilde{\in}\lambda=cl\lambda$, since λ is closed.
i.e., $\lambda\in G_{\alpha_y}$ but $\lambda\notin G_{\alpha_x}$.
Thus $G_{\alpha_x}\neq G_{\alpha_y}$. Therefore the condition holds.
Conversely let the condition hold.
Let $x,y\in X$ such that $x\neq y$ and $\alpha\in(0,1]$.
Therefore $G_{\alpha_x}\neq G_{\alpha_y}$.
Thus there exists $\lambda_\alpha\in G_{\alpha_x}$ such that $\lambda_\alpha\notin G_{\alpha_y}$ or there exists $\mu_\alpha\in G_{\alpha_y}$ such that $\mu_\alpha\notin G_{\alpha_x}$.
Therefore there exists $\lambda_\alpha\in I^X$ such that $\alpha_x\tilde{\in}cl\lambda_\alpha,\alpha_y\ \tilde{\not\in}cl\lambda_\alpha$ or there exists $\mu_\alpha\in I^X$ such that $\alpha_y\tilde{\in}cl\mu_\alpha,\alpha_x\ \tilde{\not\in}cl\mu_\alpha$.
Taking $cl\lambda_\alpha=\gamma_\alpha$ and $cl\mu_\alpha=\delta_\alpha$ we have $\exists\gamma_\alpha\in R_{\alpha_y},\gamma_\alpha\notin R_{\alpha_x}$ or $\exists\delta_\alpha\in R_{\alpha_x},\delta_\alpha\notin R_{\alpha_y}$.
Therefore (X,u) is RT_0. This completes the proof.

Theorem 3.11 *Let (X,u) be a topological space of fuzzy sets. Then $\forall x\in X$ and $\forall\alpha\in(0,1], G_{\alpha_x}$ is a proper c- grill of fuzzy sets in (X,u).*

Proof. Let $x \in X$ and $\alpha \in (0,1]$. Clearly $\tilde{0}_X \notin G_{\alpha_x}$.
Let $\lambda, \mu \in I^X$. Then

$$\lambda \geq \mu \in G_{\alpha_x} \Rightarrow \alpha_x \tilde{\in} cl\mu \leq cl\lambda \Rightarrow \lambda \in G_{\alpha_x}$$

and

$$\lambda \vee \mu \in G_{\alpha_x} \Rightarrow \alpha_x \tilde{\in} cl(\lambda \vee \mu) \Rightarrow \alpha_x \tilde{\in} (cl\lambda \vee cl\mu)$$
$$\Rightarrow \alpha_x \tilde{\in} cl\lambda \text{ or } \alpha_x \tilde{\in} cl\mu \Rightarrow \lambda \in G_{\alpha_x}, \text{ or } \ \mu \in G_{\alpha_x}.$$

Thus G_{α_x} is a grill of fuzzy sets on X.
Let $\lambda \in I^X$. Then

$$cl\lambda \in G_{\alpha_x} \Rightarrow \alpha_x \tilde{\in} cl(cl\lambda) \Rightarrow \alpha_x \tilde{\in} cl\lambda \Rightarrow \lambda \in G_{\alpha_x}.$$

Therefore G_{α_x} is a c-grill of fuzzy sets in (X,u).
Clearly $\tilde{1}_X \in G_{\alpha_x}$. Therefore $G_{\alpha_x} \neq \phi$ and hence G_{α_x} is proper.
Thus for each $x \in X$ and for each $\alpha \in (0,1]$, G_{α_x} is a proper c-grill of fuzzy sets in (X,u).

Definition 3.12 *Let $E = (\eta, (Y,v))$ be an extension of (X,u). Let $y \in Y$ and $\alpha \in (0,1]$. Define the trace $T_{(\alpha_y,E)}$ of the point α_y with respect to the extension E by*

$$T_{(\alpha_y,E)} = \{\lambda \in I^X : \alpha_y \tilde{\in} cl_v \eta(\lambda)\}.$$

When there is no chance of confusion, we shall simply write T_{α_y} for $T_{(\alpha_y,E)}$.
The α-graded trace system X^E_α of the extension E is defined by

$$X^E_\alpha = \{T_{\alpha_y} : y \in Y\}.$$

Also define $X^E_{(0,1]}$ by

$$X^E_{(0,1]} = \{T_{\alpha_y} : y \in Y, \alpha \in (0,1]\}.$$

Theorem 3.13 *Let $E = (\eta, (Y,v))$ be an extension of (X,u). Then*
(i) T_{α_y} is a proper c-grill of fuzzy sets in $(X,u), \forall y \in Y, \ \ \forall \alpha \in (0,1]$.
(ii) $T_{\eta(\alpha_x)} = G_{\alpha_x}, \forall x \in X, \ \ \forall \alpha \in (0,1]$.

Proof. (i) Let $y \in Y$ and $\alpha \in (0,1]$. Clearly $\tilde{0}_X \notin T_{\alpha_y}$.
Let $\lambda,\ \mu \in I^X$ such that $\lambda \geq \mu \in T_{\alpha_y}$. Then

$$\alpha_y \tilde{\in} cl_v \eta(\mu) \leq cl_v \eta(\lambda).$$

Therefore

$$\alpha \leq cl_v \eta(\mu)(y) \leq cl_v \eta(\lambda)(y) \Rightarrow \alpha_y \tilde{\in} cl_v \eta(\lambda) \Rightarrow \lambda \in T_{\alpha_y}.$$

$\forall \lambda, \mu \in I^X$,

$$\lambda \vee \mu \in T_{\alpha_y} \Rightarrow \alpha_y \tilde{\in} cl_v \eta(\lambda \vee \mu) \Rightarrow \alpha_y \tilde{\in} cl_v(\eta(\lambda) \vee \eta(\mu))$$
$$\Rightarrow \alpha_y \tilde{\in} cl_v \eta(\lambda) \vee cl_v \eta(\mu) \Rightarrow \alpha \leq cl_v \eta(\lambda)(y) \vee cl_v \eta(\mu)(y)$$
$$\Rightarrow \alpha \leq cl_v \eta(\lambda)(y) \text{ or } \alpha \leq cl_v \eta(\mu)(y)$$
$$\Rightarrow \alpha_y \tilde{\in} cl_v \eta(\lambda) \text{ or } \alpha_y \tilde{\in} cl_v \eta(\mu) \Rightarrow \lambda \in T_{\alpha_y} \text{ or } \mu \in T_{\alpha_y}.$$

Also for $\lambda \in I^X$,

$$cl_u \lambda \in T_{\alpha_y} \Rightarrow \alpha_y \tilde{\in} cl_v \eta(cl_u \lambda)$$
$$\Rightarrow \alpha \leq cl_v \eta(cl_u \lambda)(y) \leq cl_v(cl_v \eta(\lambda))(y),$$
$$\text{since } \eta(cl_u \lambda) = (cl_v \eta(\lambda)) \wedge \eta(\tilde{1}_X) \leq cl_v \eta(\lambda).$$
$$\Rightarrow \alpha \leq cl_v \eta(\lambda)(y) \Rightarrow \alpha_y \tilde{\in} cl_v \eta(\lambda) \Rightarrow \lambda \in T_{\alpha_y}.$$

Clearly $\tilde{1}_X \in T_{\alpha_y}$. Therefore $T_{\alpha_y} \neq \phi$.
Thus T_{α_y} is a proper c-grill of fuzzy sets in (X, u), for each $y \in Y$ and for each $\alpha \in (0, 1]$.
(ii) Let $x \in X$ and $\alpha \in (0, 1]$. Let $\lambda \in I^X$. Then

$$\lambda \in T_{\eta(\alpha_x)} \Leftrightarrow \eta(\alpha_x)\tilde{\in} cl_v\eta(\lambda) \Leftrightarrow \alpha_{\eta(x)}\tilde{\in} cl_v\eta(\lambda)$$
$$\Leftrightarrow \alpha \leq (cl_v\eta(\lambda))(\eta(x)) \Leftrightarrow \alpha \leq (cl_v\eta(\lambda) \wedge 1_{\eta(X)})(\eta(x))$$
$$\Leftrightarrow \alpha \leq (cl_v\eta(\lambda) \wedge \eta(1_X))(\eta(x)) \Leftrightarrow \alpha \leq \eta(cl_u\lambda)(\eta(x))$$
$$\Leftrightarrow \alpha \leq cl_u\lambda(x) \text{ (since } \eta \text{ is one-one)} \Leftrightarrow \lambda \in G_{\alpha_x}.$$

Thus $T_{\eta(\alpha_x)} = G_{\alpha_x}$.

Theorem 3.14 *If E_1 and E_2 be two equivalent extensions of (X, u), then $X_\alpha^{E_1} = X_\alpha^{E_2}$ for each $\alpha \in (0, 1]$ and hence $X_{(0,1]}^{E_1} = X_{(0,1]}^{E_2}$.*

Proof. Let $E_1 = (\eta_1, (Y_1, v_1))$ and $E_2 = (\eta_2, (Y_2, v_2))$ be two equivalent extensions of (X, u).
Then $\exists$ a homeomorphism h of (Y_1, v_1) onto (Y_2, v_2) such that $ho\eta_1 = \eta_2$.
Let $y \in Y_1, \alpha \in (0, 1]$ and $\lambda \in I^X$. Then

$$\lambda \in T_{(\alpha_y,E_1)} \Leftrightarrow \alpha_y\tilde{\in} cl_{v_1}\eta_1(\lambda) \Leftrightarrow h(\alpha_y)\tilde{\in} h(cl_{v_1}\eta_1(\lambda))$$
$$\Leftrightarrow \alpha_{h(y)}\tilde{\in} cl_{v_2}h(\eta_1(\lambda))$$
$$\Leftrightarrow \alpha_{h(y)}\tilde{\in} cl_{v_2}\eta_2(\lambda), \text{ since } h(\eta_1(\lambda)) = ho\eta_1(\lambda) = \eta_2(\lambda).$$
$$\Leftrightarrow \lambda \in T_{(\alpha_{(h(y)}, E_2)}.$$

Thus $T_{(\alpha_y,E_1)} = T_{(\alpha_{h(y)},E_2)}$.

$$\text{Therefore } X_\alpha^{E_1} = \{T_{(\alpha_y,E_1)} : y \in Y_1\}$$
$$= \{T_{(\alpha_{h(y)},E_2)} : y \in Y_1\}$$
$$= \{T_{(\alpha_y,E_2)} : y \in Y_2\}, \text{ because } Y_2 = \{h(y) : y \in Y_1\}.$$
$$= X_\alpha^{E_2} \quad \forall \alpha \in (0, 1].$$

Also

$$X_{(0,1]}^{E_1} = \{T_{(\alpha_y,E_1)} : y \in Y_1, \alpha \in (0, 1]\}$$
$$= \{T_{(\alpha_{h(y)},E_2)} : y \in Y_1, \alpha \in (0, 1]\}$$
$$= \{T_{(\alpha_y,E_2)} : y \in Y_2, \alpha \in (0, 1]\}$$
$$= X_{(0,1]}^{E_2}.$$

Note 3.15 *Example is given below to show that the converse of Theorem 3.14 does not hold.*

Example 3.16 *Let X, Y, Z be three infinite sets such that $X \subset Y \subset Z$ and $|X| < |Y| < |Z|$, where $|X|$ denotes the cardinal number of the set X.*
Let $u \subset I^Z$ be defined by

$$\forall \lambda \in I^Z, \lambda \in u \text{ if and only if } \lambda = \tilde{0}_Z \text{ or } Z(\lambda) \text{ is finite.}$$

Then it is clear that u is a topology of fuzzy sets on Z.
Let (X, u_X) and (Y, u_Y) be subspaces of (Z, u). Let $i : X \to Z$ be the inclusion map. Let i also denote the inclusion map of X into Y.

Obviously $E_1 = (i, (Z, u))$ is an extension of (X, u_X) and $E_2 = (i, (Y, u_Y))$ is also an extension of (X, u_X).
Note that for each $x \in X$ and $\forall \alpha \in (0, 1]$, $T_{(i(\alpha_x),E_1)} = G_{\alpha_x} = T_{(i(\alpha_x),E_2)}$, i.e., $T_{(\alpha_x,E_1)} = G_{\alpha_x} = T_{(\alpha_x,E_2)}$.
Let $G^* = \{\lambda \in I^X : \lambda(a) = 1$ for infinitely many points a of $X\}$.
Then it is easy to check that $\forall \alpha \in (0, 1]$,

$$T_{(\alpha_z,E_1)} = G^*, \quad \forall z \in Z - X \text{ and } T_{(\alpha_y,E_2)} = G^*, \quad \forall y \in Y - X.$$

Hence $X^{E_1}_\alpha = X^{E_2}_\alpha$, $\forall \alpha \in (0, 1]$ and hence $X^{E_1}_{(0,1]} = X^{E_2}_{(0,1]}$.
But $E_1 \not\approx E_2$, as $|Y| < |Z|$.

Theorem 3.17 *For any extension $E = (\eta, (Y, v))$ of (X, u) and $\forall y, z \in Y$, $G_{\alpha_y} \subset G_{\alpha_z}$ implies $T_{\alpha_y} \subset T_{\alpha_z}$ for each $\alpha \in (0, 1]$.*

Proof. Let $\alpha \in (0, 1]$ and $y, z \in Y$ be such that $G_{\alpha_y} \subset G_{\alpha_z}$.
Then $\forall \mu \in I^X$,

$$\mu \in T_{\alpha_y} \Rightarrow \alpha_y \tilde{\in} cl_v \eta(\mu) \Rightarrow \eta(\mu) \in G_{\alpha_y} \Rightarrow \eta(\mu) \in G_{\alpha_z}, \text{ since } G_{\alpha_y} \subset G_{\alpha_z}$$
$$\Rightarrow \alpha_z \tilde{\in} cl_v \eta(\mu) \Rightarrow \mu \in T_{\alpha_z}.$$

Thus $T_{\alpha_y} \subset T_{\alpha_z}$.

Note 3.18 *An example is given below to show that the converse of the above theorem is not true.*

Example 3.19 *Let Y be an infinite set. Let $v \subset I^Y$ be defined by*

$$\forall \lambda \in I^Y, \lambda \in v \quad \text{if and only if} \quad \lambda = \tilde{0}_Y \text{ or } Z(\lambda) \text{ is finite.}$$

Clearly v is a topology of fuzzy sets on Y.
Let X be an infinite set such that $X \subset Y$ and $|Y - X| \geq 2$ and $i : X \to Y$ be the inclusion map. Then it is easy to check that $(i, (Y, v))$ is an extension of (X, v_X).
Let $y, z(\neq y) \in Y - X$. Then it is clear that

$$T_{\alpha_y} = \{\lambda \in I^X : \lambda(a) = 1 \text{ for infinitely many points } a \text{ of } X\}$$
$$= T_{\alpha_z}, \quad \forall \alpha \in (0, 1].$$

Choose λ, $\mu \in I^Y$ such that

$$\lambda(y) = 0.5,\ \lambda(z) = 0.6,\ \mu(y) = 0.6,\ \mu(z) = 0.3$$

and both the sets $\{a \in Y : \lambda(a) = 1\}$ and $\{a \in Y : \mu(a) = 1\}$ are finite.
Then it is clear that $\lambda \in G_{0.6_z}$, $\lambda \notin G_{0.6_y}$ and $\mu \in G_{0.6_y}$, $\mu \notin G_{0.6_z}$.
Thus $G_{0.6_z} \not\subset G_{0.6_y}$ and $G_{0.6_y} \not\subset G_{0.6_z}$.

However the following result holds.

Theorem 3.20 *If $(\eta, (Y, v))$ is a principal extension of (X, u), then $\forall y, z \in Y$,*

$$T_{\alpha_y} \subset T_{\alpha_z} \text{ if and only if } G_{\alpha_y} \subset G_{\alpha_z} \text{ for each } \alpha \in (0, 1].$$

Proof. 'If part' has already been proved above.
Let $\alpha \in (0,1]$ and $y, z \in Y$ such that $T_{\alpha_y} \subset T_{\alpha_z}$.
Let $\lambda \in I^Y$ such that $\lambda \in G_{\alpha_y}$. Then $\alpha_y \tilde{\in} cl_v\lambda$.
Since $\{\ cl_v\eta(\mu) : \mu \in I^X\ \}$ is a base for the closed sets in (Y, v),

$\alpha_y \tilde{\in}\ \wedge \{\ cl_v\eta(\mu) : \mu \in I^X,\ cl_v\eta(\mu) \geq \lambda\ \}$.

Thus

$\alpha_y \tilde{\in}\ cl_v\eta(\mu),\ \forall \mu \in I^X$ with $cl_v\eta(\mu) \geq \lambda$,

and hence

$\mu \in T_{\alpha_y},\ \forall \mu \in I^X$ with $cl_v\eta(\mu) \geq \lambda$.

Since $T_{\alpha_y} \subset T_{\alpha_z},\quad \mu \in T_{\alpha_z}\quad \forall \mu \in I^X$ with $cl_v\eta(\mu) \geq \lambda$, which implies that

$\alpha_z \tilde{\in}\ \wedge \{\ cl_v\eta(\mu) : \mu \in I^X,\ cl_v\eta(\mu) \geq \lambda\ \}$.

i.e., $\alpha_z \tilde{\in}\ cl_v\lambda$ i.e., $\lambda \in G_{\alpha_z}$.
Hence $G_{\alpha_y} \subset G_{\alpha_z}$.

The following corollary is an easy consequence of the above theorem.

Corollary 3.21 *If $(\eta, (Y, v))$ is a principal extension of (X, u), then $\forall y, z \in Y,\ T_{\alpha_y} = T_{\alpha_z}$ if and only if $G_{\alpha_y} = G_{\alpha_z}$ for each $\alpha \in (0,1]$.*

Theorem 3.22 *If $(\eta, (Y, v))$ is a principal extension of (X, u), then (Y, v) is RT_0 if and only if*

$\forall y, z \in Y, T_{\alpha_y} = T_{\alpha_z}$ for some $\alpha \in (0,1] \Rightarrow y = z$.

Proof. Let (Y, v) be RT_0. Let $y, z \in Y$ such that $T_{\alpha_y} = T_{\alpha_z}$ for some $\alpha \in (0,1]$.
Thus $G_{\alpha_y} = G_{\alpha_z}$ and hence $y = z$ (see Theorem 3.10).
Conversely suppose that the condition holds.
i.e., $\forall y, z \in Y, T_{\alpha_y} = T_{\alpha_z}$ for some $\alpha \in (0,1]$ implies $y = z$.
Let $G_{\alpha_y} = G_{\alpha_z}$ for some $\alpha \in (0,1]$. Therefore by the above corollary we have $T_{\alpha_y} = T_{\alpha_z}$ and hence by the given condition we have $y = z$.
Hence (Y, v) is RT_0 (see Theorem 3.10).

4 Construction of RT_0 Principal Extension of an RT_0 Topological Space with the Given α-graded Trace System

In this section (X, u) will be an RT_0 topological space of fuzzy sets and for each $\alpha \in (0,1]$, X^*_α be a collection of proper c-grills of fuzzy sets in (X, u) such that $G_{\alpha_x} \in X^*_\alpha, \forall x \in X$.
Let $\alpha \in (0,1]$. Define,

$f_\alpha : X \to X^*_\alpha$ by $f_\alpha(x) = G_{\alpha_x}, \forall x \in X$.

In view of Theorem 3.10, it follows that f_α is one-one.
$\forall \lambda \in I^X$, , define $\lambda^c_\alpha : X^*_\alpha \to I$ by the following :

$$\lambda^c_\alpha(G_{\alpha_x}) = cl_u\lambda(x), \forall x \in X$$

and for $G \in X^*_\alpha - \{G_{\alpha_x} : x \in X\}$,

$$\lambda^c_\alpha(G) = \begin{cases} 1 & \text{if} \quad \lambda \in G \\ 0 & \text{if} \quad \lambda \notin G. \end{cases}$$

Let $\lambda, \mu \in I^X$. $\forall x \in X$

$$(\lambda \vee \mu)^c_\alpha(G_{\alpha_x}) = cl_u(\lambda \vee \mu)(x) = (cl_u\lambda \vee cl_u\mu)(x) = cl_u\lambda(x) \vee cl_u\mu(x)$$
$$= \lambda^c_\alpha(G_{\alpha_x}) \vee \mu^c_\alpha(G_{\alpha_x}) \quad = (\lambda^c_\alpha \vee \mu^c_\alpha)(G_{\alpha_x}).$$

Also for $G \in X^*_\alpha - \{G_{\alpha_x} : x \in X\}$,

$$(\lambda \vee \mu)^c_\alpha(G) = (\lambda^c_\alpha \vee \mu^c_\alpha)(G) \; ,$$

since $\lambda \vee \mu \in G$ if and only if $\lambda \in G$ or $\mu \in G$.

Thus $(\lambda \vee \mu)^c_\alpha = \lambda^c_\alpha \vee \mu^c_\alpha, \forall \lambda, \mu \in I^X$. Also $(\tilde{0}_X)^c_\alpha = \tilde{0}_{X^*_\alpha}$.
Thus $\{\lambda^c_\alpha : \lambda \in I^X\}$ is a base for the closed sets of a topology w_α (say) of fuzzy sets on X^*_α.

Theorem 4.1 *Let $\alpha \in (0,1]$ and $(X,u), (X^*_\alpha, w_\alpha)$ and the other symbols used below be same as above. Then*
(i) $\forall \lambda, \mu \in I^X, \lambda \leq \mu \Rightarrow \lambda^c_\alpha \leq \mu^c_\alpha$.
(ii) $\forall \lambda \in I^X, (cl_u\lambda)^c_\alpha = \lambda^c_\alpha$.
(iii) $\forall \lambda, \mu \in I^X, f_\alpha(\lambda) \leq \mu^c_\alpha \Leftrightarrow cl_u\lambda \leq cl_u\mu$.
(iv) $\forall \lambda \in I^X, cl_{w_\alpha} f_\alpha(\lambda) = \lambda^c_\alpha$.
(v) $cl_{w_\alpha} f_\alpha(\tilde{1}_X) = \tilde{1}_{X^*_\alpha}$.
(vi) $\forall \lambda \in I^X, (\; cl_{w_\alpha} f_\alpha(\lambda) \;) \wedge f_\alpha(\tilde{1}_X) = f_\alpha(cl_u\lambda)$.

Proof. Let $\alpha \in (0,1]$.
(i) $\forall \lambda, \mu \in I^X$,

$$\lambda \leq \mu \Rightarrow \lambda^c_\alpha(G) \leq \mu^c_\alpha(G), \forall G \in X^*_\alpha \Rightarrow \lambda^c_\alpha \leq \mu^c_\alpha.$$

(ii) Let $\lambda \in I^X$. Then

$$(cl_u\lambda)^c_\alpha(G_{\alpha_x}) = cl_u(cl_u\lambda)(x) = cl_u\lambda(x) = \lambda^c_\alpha(G_{\alpha_x}), \forall x \in X$$

and clearly

$$(cl_u\lambda))^c_\alpha(G) = \lambda^c_\alpha(G) \text{ if } G \in X^*_\alpha - \{G_{\alpha_x} : x \in X\},$$

since G is a c-grill of fuzzy sets in X.

Thus $(cl_u\lambda)^c_\alpha(G) = \lambda^c_\alpha(G), \forall G \in X^*_\alpha$.
Hence $(cl_u\lambda)^c_\alpha = \lambda^c_\alpha, \forall \lambda \in I^X$.
(iii) For $\lambda, \mu \in I^X$,

$$\begin{aligned} f_\alpha(\lambda) \leq \mu^c_\alpha &\Leftrightarrow f_\alpha(\lambda)(G) \leq \mu^c_\alpha(G), \forall G \in X^*_\alpha \\ &\Leftrightarrow f_\alpha(\lambda)(G_{\alpha_x}) \leq \mu^c_\alpha(G_{\alpha_x}), \forall x \in X \\ &\Leftrightarrow f_\alpha(\lambda)(f_\alpha(x)) \leq cl_u\mu(x), \forall x \in X \\ &\Leftrightarrow \lambda(x) \leq cl_u\mu(x), \forall x \in X, \text{ since } f_\alpha \text{ is one-one.} \\ &\Leftrightarrow \lambda \leq cl_u\mu \\ &\Leftrightarrow cl_u\lambda \leq cl_u\mu. \end{aligned}$$

(iv) $\forall \lambda \in I^X$,

$cl_{w_\alpha} f_\alpha(\lambda) = \wedge\{\ \mu_\alpha^c : \mu_\alpha^c \geq f_\alpha(\lambda),\ \mu \in I^X\}$, since $\{\mu_\alpha^c : \mu \in I^X\}$ is a base for the closed sets in (X_α^*, w_α).

$= \wedge\{\ \mu_\alpha^c : cl_u\lambda \leq cl_u\mu, \mu \in I^X\ \}$
$= \wedge\{\ (cl_u\mu)_\alpha^c : cl_u\lambda \leq cl_u\mu, \mu \in I^X\ \}$
$= (cl_u\lambda)_\alpha^c$
$= \lambda_\alpha^c$.

(v) $cl_{w_\alpha} f_\alpha(\tilde{1}_X) = (\tilde{1}_X)_\alpha^c = \tilde{1}_{X_\alpha^*}$, since $(\tilde{1}_X)_\alpha^c(G) = 1 = \tilde{1}_{X_\alpha^*}(G)\ \forall G \in X_\alpha^*$.
(vi) Let $\lambda \in I^X$. Then $\forall x \in X$,

$\Big(\big(\ cl_{w_\alpha} f_\alpha(\lambda)\ \big) \wedge f_\alpha(\tilde{1}_X)\Big)(G_{\alpha_x}) = \Big(\lambda_\alpha^c \wedge f_\alpha(\tilde{1}_X)\Big)(G_{\alpha_x})$
$= cl_u\lambda(x) \wedge \tilde{1}_X(x) = cl_u\lambda(x) = f_\alpha(cl_u\lambda)(f_\alpha(x))$(since f_α is one-one)
$= f_\alpha(cl_u\lambda)(G_{\alpha_x})$.

Also if $G \in X_\alpha^* - \{G_{\alpha_x} : x \in X\}$, then

$\Big(\big(\ cl_{w_\alpha} f_\alpha(\lambda)\ \big) \wedge f_\alpha(\tilde{1}_X)\Big)(G) = \Big(\lambda_\alpha^c \wedge f_\alpha(\tilde{1}_X)\Big)(G) = \lambda_\alpha^c(G) \wedge 0$
$= 0 = f_\alpha(cl_u\lambda)(G)$.

Thus $(\ cl_{w_\alpha} f_\alpha(\lambda)\ ) \wedge f_\alpha(\tilde{1}_X) = f_\alpha(cl_u\lambda)$.
This completes the proof.

Remark 4.2 *Since for each* $\alpha \in (0,1]$, $f_\alpha : X \to X_\alpha^*$ *is one-one and* $\forall \lambda \in I^X, (\ cl_{w_\alpha} f_\alpha(\lambda)\ ) \wedge f_\alpha(\tilde{1}_X) = f_\alpha(cl_u\lambda)$ *and* $cl_{w_\alpha} f_\alpha(\tilde{1}_X) = \tilde{1}_{X_\alpha^*}$, *it follows that* $(f_\alpha, (X_\alpha^*, w_\alpha))$ *is an extension of* (X, u) *for each* $\alpha \in (0,1]$.
Since for each $\alpha \in (0,1]$, $\{\ \lambda_\alpha^c : \lambda \in I^X\}$ *is a base for the closed sets of* (X_α^*, w_α) *and* $cl_{w_\alpha} f_\alpha(\lambda) = \lambda_\alpha^c, \forall \lambda \in I^X$, *it follows that* $(f_\alpha, (X_\alpha^*, w_\alpha))$ *is a principal extension of* (X, u) *for each* $\alpha \in (0,1]$.
Note that $\forall G_{\alpha_x} \in X_\alpha^*$,

$T_{\alpha_{G_{\alpha_x}}} = \{\ \mu \in I^X : \alpha_{G_{\alpha_x}} \tilde{\in}\ cl_{w_\alpha} f_\alpha(\mu)\ \}$
$= \{\mu \in I^X : (\ cl_{w_\alpha} f_\alpha(\mu)\ )(G_{\alpha_x}) \geq \alpha\ \}$
$= \{\ \mu \in I^X : \mu_\alpha^c(G_{\alpha_x}) \geq \alpha\ \}$
$= \{\ \mu \in I^X : cl_u\mu(x) \geq \alpha\ \}$
$= \{\ \mu \in I^X : \alpha_x \tilde{\in}\ cl_u\mu\ \}$
$= G_{\alpha_x}$.

Also if $G \in X_\alpha^* - \{G_{\alpha_x} : x \in X\}$, *then*

$T_{\alpha_G} = \{\mu \in I^X : \mu_\alpha^c(G) \geq \alpha\}$
$= \{\mu \in I^X : \mu_\alpha^c(G) = 1\}$
$= \{\mu \in I^X : \mu \in G\}$
$= G$.

Thus $T_{\alpha_G} = G,\ \ \forall G \in X_\alpha^*$.
Therefore X_α^* *is the* α*-graded trace system of the extension* $(f_\alpha, (X_\alpha^*, w_\alpha))$.
Also for each $\alpha \in (0,1]$ *we have,*

$\forall G_1, G_2 \in X_\alpha^*, T_{\alpha_{G_1}} = T_{\alpha_{G_2}} \Rightarrow G_1 = G_2$,

and hence (X_α^*, w_α) *is* RT_0 *for each* $\alpha \in (0,1]$.
Thus $(f_\alpha, (X_\alpha^*, w_\alpha))$ *is an* RT_0 *principal extension of* (X, u) *with the given* α*-graded trace system for each* $\alpha \in (0,1]$.

Notation 4.3 *The extension $(f_\alpha, (X^*_\alpha, w_\alpha))$ will be denoted by $E_\alpha(X^*_\alpha)$. Thus $X_\alpha^{E_\alpha(X^*_\alpha)} = X^*_\alpha$.*

5 Future Work

In [6], we introduced T_0 principal extensions of a T_0-topological spaces of fuzzy sets. In [8], we defined fuzzy conjoint compactness and fuzzy linkage compactness and established conditions on the trace systems which would ensure the fuzzy conjoint compactness and fuzzy linkage compactness of the T_0 principal fuzzy extensions. In [8], we also introduced basic fuzzy proximities, Lodato fuzzy proximities and eventually proved a theorem which establishes that there is a bijection between a class of Lodato fuzzy proximities compatible with a given strongly T_1- topological space of fuzzy sets (X, c) and the class of strongly T_1 principal Type-II fuzzy linkage compactifications of (X, c). Our aim is to achieve the similar result mentioned above in the RT_0 spaces.

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