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Weak Law of Large Numbers for Hybrid Variables Based on Chance Measure

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Abstract

Based upon previous studies on laws of large numbers for fuzzy, random, fuzzy random and random fuzzy variables, We go further to explore weak law of large numbers(WLLN) for hybrid variables comprising fuzzy random variables and random fuzzy variables. we mainly prove Chebyshev WLLN, Poisson WLLN, Bernoulli WLLN, Markov WLLN and Khintchin WLLN for hybrid variables based on chance measure.

Keywords: *weak law of large numbers(WLLN), fuzzy random variable, random fuzzy variable, hybrid variable, chance measure.*

1 Introduction

The concept of fuzzy random variables was first promoted by Kwakernaak[1][2], and developed by Puri and Ralescu[3][4], Stein and Talati[5], Kruse and Meyer[6], Liu and Liu[7], and so on. In addition, a random fuzzy variable was introduced by Liu[8]. More generally, a hybrid variable was proposed by Liu[9] as a measurable function from a chance space to the set of real numbers.

Theories for hybrid variable has been applied in a lot of fields, such as the random fuzzy economic manufacturing quantity model[10], the confidence-interval-based fuzzy random regression models[11], the fuzzy random decision systems[12], the project scheduling problem with random fuzzy activity duration times[13], chance-constrained goal programming models with random and fuzzy parameters[14].

Based on possibility measure[15][16], the strong law of large numbers (SLLN) was respectively proved by several researchers such as Kruse[17], Miyakoshi and Shimbo[18], Taylor, Daffer and Patterson[19], Klement, Puri and Ralescu [20], Inoue[21], Kim, Joo, Kim and Kwon[22][23][24], Li and Ogura [25], Proske and Puri[26],and so on. Taylor, Seymour and Chen [27] proved WLLN for fuzzy random sets. Chen and Liu[28] studied the law of large numbers for fuzzy variables based on credibility measure. Liu and Liu[7] studied SLLN and WLLN for fuzzy random variables based on chance measure.

To sum up, many studies have been done on the WLLN and SLLN for fuzzy, random or fuzzy random variables, nevertheless, there is little researcher to study WLLN or SLLN for random fuzzy or hybrid variables based on chance measure.

The present paper is concerned with the research of Chebyshev WLLN, Poisson WLLN, Bernoulli WLLN, Markov WLLN and Khintchin WLLN for hybrid variables based on chance measure, which should be helpful to develop further the theory and its applications of hybrid variables.

2 Preliminaries

In this section, we describe some concepts and results of hybrid variables based on chance measure.

Definition 2.1 (*Liu*[9]) *Suppose that* (Θ, P, Cr) *is a credibility space and* (Ω, A, Pr) *is a probability space. The product* $(\Theta, \mathrm{P}, Cr) \times (\Omega, \mathrm{A}, Pr)$ *is called a chance space.*

Definition 2.2 (*Liu*[9]) *A hybrid variable is a measurable function from a chance space* $(\Theta, \mathrm{P}, Cr) \times (\Omega, \mathrm{A}, Pr)$ *to the set of real numbers, i.e., for any Borel set B of real numbers, the set*

$$\{\xi \in B\} = \{\theta, \omega\} \in \Theta \times \Omega \mid \xi(\theta, \omega) \in B\}$$

is an event.

Definition 2.3 (*Liu*[29]) *The chance distribution* $\Phi : R \to [0,1]$ *of a hybrid variable* ξ *is defined by*

$$\Phi(x) = Ch\{(\theta, \omega) \in \Theta \times \Omega \mid \xi(\theta, \omega) \le x\}.$$

Definition 2.4 (*Liu*[29]) *The hybrid variables* $\xi_1, \xi_2, \cdots, \xi_n$ *are said to be independent if*

$$E\left[\sum_{i=1}^{n} f_i(\xi_i)\right] = \sum_{i=1}^{n} E[f_i(\xi_i)]$$

for any measurable functions $f_1, f_2, \cdots, f_n$ *provided that the expected values exist and are finite.*

Definition 2.5 (Liu[29]) *The hybrid variables* ξ *and* η *are identically distributed if*

$$Ch\{\xi \in B\} = Ch\{\eta \in B\}$$

for any Borel set B of real numbers.

Definition 2.6 (*Liu*[29]) *Let* ξ *be a hybrid variable, then the expected variable of* ξ *is defined by*

$$E[\xi] = \int_0^{+\infty} Ch\{\xi \geq r\}dr - \int_{-\infty}^0 Ch\{\xi \leq r\}dr$$

provided that at least one of the two integrals is definite.

Definition 2.7 (*Liu* [29]) *Let* ξ *be a hybrid variable with finite expected value* e*, then the variable of* ξ *is defined by*

$$V[\xi] = E[(\xi - e)^2].$$

Theorem 2.8 (*Liu* [29]) *If* ξ *and* η *are independent hybrid variables with finite expected values, then we have*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta].$$

for any real numbers a and b.

Theorem 2.9 (*Liu* [29] *Chebyshev Inequality*) *Let* ξ *be a hybrid variable whose variance* $V[\xi]$ *exists, then for any given number* $t > 0$*, we have*

$$Ch\{|\xi - E[\xi]| \geq t\} \leq \frac{V[\xi]}{t^2}.$$

Theorem 2.10 (*Liu* [29] *Markov Inequality*) *Let* ξ *be a hybrid variable, then for any given numbers* $t > 0$ *and* $p > 0$*, we have*

$$Ch\{|\xi| \geq t\} \leq \frac{E[|\xi|^p]}{t^p}.$$

Theorem 2.11 (*Li and Liu* [30] *Chance Subadditivity Theorem*) *The chance measure is subadditive. That is*

$$Ch\{\Lambda_1 \cup \Lambda_2\} \leq Ch\{\Lambda_1\} \cup Ch\{\Lambda_2\}.$$

for any events Λ_1 and Λ_2.

Theorem 2.12 (*Li and Liu* [30]) *The chance measure is self-dual. That is*

$$Ch\{\Lambda\} + Ch\{\Lambda^c\} = 1.$$

Theorem 2.13 (*Liu*[29]) *Let* $(\Theta, \mathrm{P}, Cr) \times (\Omega, \mathrm{A}, Pr)$ *be a chance space and Ch a chance measure., then we have*

$$Ch\{\varnothing\} = 0, \quad Ch\{\Theta \times \Omega\} = 1, \quad 0 \leq Ch\{\Lambda\} \leq 1.$$

for any event Λ.

Theorem 2.14 (*Li and Liu* [30]) *The chance measure is increasing. That is,*

$$Ch\{\Lambda_1\} \leq Ch\{\Lambda_2\}.$$

for any event Λ_1 *and* Λ_2 *with* $\Lambda_1 \subset \Lambda_2$.

3 The WLLN for hybrid variables based on chance measure

In this section, we attempt to discuss WLLN, Chebyshev WLLN, Poisson WLLN, Bernoulli WLLN, Markov WLLN and Khintchin WLLN.

Definition 3.1 *Suppose that* $\{\xi_i\}$ *is a sequence of hybrid variables whose expected value* $E[\xi_i]$ *exists, and its average* $S_n = \frac{1}{n}\sum_{i=1}^{n}\xi_i (i = 1, 2, \cdots, n)$ *is finite. For any given* $\varepsilon > 0$, *if* $\lim_{n\to\infty} Ch\{|S_n - E[S_n]| \geq \varepsilon\} = 0$, *we say that* $\{\xi_i\}$ *satisfies the WLLN.*

Lemma 3.2 *Let* $\xi_1, \xi_2, \cdots, \xi_n$ *be independent hybrid variables with finite expected value e, then we get*

$$E[S_n] = e$$

Proof. Obviously, by Theorem 2.8, the lemma is proved.

Lemma 3.3 *Let* ξ *and* η *be independent hybrid variables with finite expected values and variances. Then we have*

$$V[a\xi + b\eta] = a^2V[\xi] + b^2V[\eta]$$

for any real numbers a and b.

Proof. Its follows from Definition 2.7 that

$$\begin{aligned}V[a\xi + b\eta] &= E[(a\xi + b\eta - E[a\xi + b\eta])^2]\\ &= E[(a\xi + b\eta - aE[\xi] + bE[\eta])^2]\\ &= a^2V[\xi] + b^2V[\eta] + 2abE[(\xi - E[\xi])(\eta - E[\eta])]\\ &= a^2V[\xi] + b^2V[\eta] + 2abE[(\xi - E[\xi])]E[(\eta - E[\eta])]\\ &= a^2V[\xi] + b^2V[\eta]\end{aligned}$$

The lemma is proved.

Lemma 3.4 *Let* $\xi_1, \xi_2, \cdots, \xi_n$ *be independent, identically distributed (i.i.d.) hybrid variables having a finite expected value e and variance* σ^2, *then we can obtain*

$$V[S_n] = \frac{\sigma^2}{n}$$

Proof. Obviously, by Lemma 3.3, the lemma is proved.

Theorem 3.5 (*WLLN*) *Let* $\xi_1, \xi_2, \cdots, \xi_n$ *be i.i.d. hybrid variables with the same expected value e and variance* σ^2*, then we say the sequence* $\{\xi_i\}$ *satisfies WLLN.*

i.e.

$$\lim_{n\to\infty} Ch\{|S_n - e| \ge \varepsilon\} = 0$$

holds for any given $\varepsilon > 0$.

Thus the average S_n converges in chance (or weakly) to the expected value e as $n \to \infty$.

Proof. By using Lemma 3.2, 3.4 and Theorem 2.9, we obtain

$$Ch\{|S_n - e| \ge \varepsilon\} \le \frac{\sigma^2}{n\varepsilon^2}$$

for any given $\varepsilon > 0$,

i.e.

$$\lim_{n\to\infty} Ch\{|S_n - e| \ge \varepsilon\} = 0.$$

The theorem is proved.

Theorem 3.6 (*Chebyshev WLLN*) *Let* $\{\xi_i\}$ *be a sequence of independent hybrid variables and a constant* $v > 0$ such that its variance $V[\xi_i] < v$ for all $i (i = 1, 2, \cdots, n)$ *then we say* $\{\xi_i\}$ *satisfies WLLN.*

i.e.

$$\lim_{n\to\infty} Ch\{|S_n - E[S_n]| \ge \varepsilon\} = 0$$

holds for any given $\varepsilon > 0$.

Proof. For any given $\varepsilon > 0$, it follows from Theorem 2.9 and Lemma 3.4 that

$$0 \le \lim_{n\to\infty} Ch\{|S_n - E[S_n]| \ge \varepsilon\} \le \lim_{n\to\infty}\left(\frac{1}{\varepsilon^2} V[S_n]\right) \le \lim_{n\to\infty} \frac{v}{n\varepsilon^2} = 0$$

The theorem is proved.

Theorem 3.7 (*Poisson WLLN*) *Assume that* $\{\xi_i\}$ $(i = 1, 2, \cdots, n)$ *is a sequence of independent hybrid variables. For* $0 < ch_i < 1$, *if*

$Ch\{\xi_i = 1\} = ch_i$, $Ch\{\xi_i = 0\} = 1 - ch_i$,

then $\{\xi_i\}$ *satisfies WLLN,*

i.e.

$$\lim_{n\to\infty} Ch\left\{\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i - \frac{1}{n}\sum_{i=1}^{n}ch_i\right| \ge \varepsilon\right\} = 0$$

hold for any given $\varepsilon > 0$.

Proof. Since

$$Ch\{\xi_i = 1\} = ch_i, \quad Ch\{\xi_i = 0\} = 1 - ch_i,$$

by Theorem 2.12, the distribution of $\{\xi_i\}$ satisfies

ξ_i	0	1
Ch	$1-ch_i$	ch_i

And it satisfies

$$S_n = \frac{1}{n}\sum_{i=1}^{n}\xi_i, \quad E[S_n] = E\left[\frac{1}{n}\sum_{i=1}^{n}\xi_i\right] = \frac{1}{n}E\left[\sum_{i=1}^{n}\xi_i\right] = \frac{1}{n}\sum_{i=1}^{n}ch_i$$

holds.

Thus, since the sequence $\{\xi_i\}$ satisfies the Chebyshev WLLN (Theorem 3.6), for any given $\varepsilon > 0$, we obtain

$$\lim_{n\to\infty} Ch\left\{\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i - \frac{1}{n}\sum_{i=1}^{n}ch_i\right| \ge \varepsilon\right\} = \lim_{n\to\infty} Ch\left\{\left|S_n - E[S_n]\right| \ge \varepsilon\right\} = 0$$

The theorem is proved.

Theorem 3.8 (*Bernoulli WLLN) Suppose that* $\{\xi_i\}$ $(i = 1, 2, \cdots, n)$ *is a sequence of i.i.d. hybrid variables. For* $0 < ch < 1$, *if*

$$Ch\{\xi_i = 1\} = ch, \quad Ch\{\xi_i = 0\} = 1 - ch,$$

then $\{\xi_i\}$ *satisfies WLLN,*

i.e.

$$\lim_{n\to\infty} Ch\left\{\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i - ch\right| \ge \varepsilon\right\} = 0$$

hold for any given $\varepsilon > 0$.

Proof. In fact, it is a special cause of Poisson WLLN (Theorem 3.7),

i.e.

$$ch_1 = ch_2 = \cdots = ch_n = ch.$$

Since the sequence $\{\xi_i\}$ satisfies

ξ_i	0	1
Ch	$1-ch$	ch

Obviously,

$$E[\xi_i]=ch_i=ch, \quad V[\xi_i]=ch(1-ch)<v.$$

for a positive constant v.

Additionally, the sequence $\{\xi_i\}$ is *i.i.d., then*

$$\lim_{n\to\infty} Ch\left\{\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i-\frac{1}{n}\sum_{i=1}^{n}ch_i\right|\geq\varepsilon\right\}=\lim_{n\to\infty} Ch\left\{\left|S_n-E[S_n]\right|\geq\varepsilon\right\}=0$$

holds.

Thus, since the sequence $\{\xi_i\}$ satisfies the Chebyshev WLLN (Theorem 3.6), for any given $\varepsilon>0$, we obtain

$$S_n=\frac{1}{n}\sum_{i=1}^{n}\xi_i, \quad E[S_n]=E\left[\frac{1}{n}\sum_{i=1}^{n}\xi_i\right]=\frac{1}{n}E\left[\sum_{i=1}^{n}\xi_i\right]=\frac{1}{n}\sum_{i=1}^{n}ch_i=ch$$

hold.

Thus, since the sequence $\{\xi_i\}$ satisfies the Poisson WLLN (Theorem 3.7), for any given $\varepsilon>0$, we obtain

$$\lim_{n\to\infty} Ch\left\{\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i-ch\right|\geq\varepsilon\right\}=0$$

The theorem is proved.

Theorem 3.9 *(Markov WLLN) Assume that a sequence* $\{\xi_i\}$ *of hybrid variables satisfies*

$$\lim_{x\to\infty} V[S_n]=0,$$

Then we say $\{\xi_i\}$ *satisfies WLLN,*

i.e.

$$\lim_{n\to\infty} Ch\left\{\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i-E[S_n]\right|\geq\varepsilon\right\}=0 \ .$$

hold for any given $\varepsilon>0$.

Proof. For any given $\varepsilon>0$, it follows from Theorem 2.9 that

$$0 \le \lim_{n\to\infty} Ch\{|S_n - E[S_n]| \ge \varepsilon\} \le \lim_{n\to\infty}\left(\frac{1}{\varepsilon^2}V[S_n]\right) = 0$$

The theorem is proved.

Lemma 3.10 *Let* the *hybrid variables* ξ *and* η *be independent, then*

$$Ch\left\{\left(|\xi|+|\eta|\right) \ge 2\varepsilon\right\} \le Ch\left\{|\xi| \ge \varepsilon\right\} + Ch\left\{\eta \ne 0\right\}$$

hold for any given $\varepsilon > 0$.

Proof. Obviously, for any given $\varepsilon > 0$,

$$\left\{\left(|\xi|+|\eta|\right) \ge 2\varepsilon\right\} \subset \left\{|\xi| \ge \varepsilon\right\} \cup \left\{\eta \ge \varepsilon\right\} \subset \left\{|\xi| \ge \varepsilon\right\} \cup \left\{\eta \ne 0\right\}.$$

By using Theorem 2.11 and 2.14, we can have

$$Ch\left\{\left(|\xi|+|\eta|\right) \ge 2\varepsilon\right\} \subset Ch\left\{\left\{|\xi| \ge \varepsilon\right\} \cup \left\{\eta \ne 0\right\}\right\} + Ch\left\{|\xi| \ge \varepsilon\right\} + Ch\left\{\eta \ne 0\right\}.$$

The lemma is proved.

Theorem 3.11 *(Khintchin WLLN) Let* $\{\xi_i\}$ *be a sequence of i.i.d. hybrid variables with finite expected values* $E[\xi_i] = e$, $(i = 1, 2, \cdots, n)$, *then* $\{\xi_i\}$ *satisfies WLLN,*

i.e.

$$\lim_{n\to\infty} Ch\left\{|S_n - e| \ge \varepsilon\right\} = 0.$$

hold for any given $\varepsilon > 0$.

Proof. For any given $\varepsilon > 0$, we define two hybrid sequences as follows:

$$\eta_i = \xi_i \chi_{|\xi_i - e| < n\delta} = \begin{cases} \xi_i & if\ |\xi_i - e| < n\delta, \\ 0 & if\ |\xi_i - e| \ge n\delta, \end{cases}$$

$$\varsigma_i = \xi_i \chi_{|\xi_i - e| \ge n\delta} = \begin{cases} 0 & if\ |\xi_i - e| < n\delta, \\ \xi_i & if\ |\xi_i - e| \ge n\delta, \end{cases}$$

Where

$$\chi_{|\xi_i - e| \ge n\delta} = \begin{cases} 1 & if\ |\xi_i - e| < n\delta, \\ 0 & if\ |\xi_i - e| \ge n\delta, \end{cases}$$

$$i = 1, 2, \cdots, n.$$

Let

$$\begin{cases} M = E\left[|\xi_i - e|\right], \\ e_n = E[\eta_i]. \end{cases}$$

By the conditions of Theorem 3.11, we can have

(1) $\xi_i = \eta_i + \varsigma_i$.

(2) $M < +\infty$, $e_n < +\infty (n = 1, 2, \cdots)$. And for $n \to \infty$, we obtain

$$e_n = \int_0^{+\infty} Ch\left\{\xi_i \chi_{|\xi_i - e| < n\delta} \ge t\right\} dt - \int_{-\infty}^0 Ch\left\{\xi_i \chi_{|\xi_i - e| < n\delta} \le t\right\} dt \to e. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

(3)By Theorem 2.10, for $n = 1, 2, \cdots$, we can have

$$\begin{aligned} V[\eta_i] &= E\left[\left(\eta_i - e_n\right)^2\right] \\ &= \int_0^{+\infty} Ch\left\{\left|\xi_i \chi_{|\xi_i - e| < n\delta} - e_n\right|^2 \ge t\right\} dt - \int_{-\infty}^0 Ch\left\{\left|\xi_i \chi_{|\xi_i - e| < n\delta} - e_n\right|^2 \le t\right\} dt \\ &= \int_0^{(n\delta)^2} Ch\left\{\left|\xi_i - e_n\right|^2 \ge t\right\} dt \\ &\le \int_0^{(n\delta)^2} \left(\frac{E\left[\xi_i - e_n\right]}{\sqrt{t}}\right) dt \\ &\le 2n\delta M. \end{aligned}$$

Thus,

$$V\left[\frac{1}{n}\sum_{i=1}^n \eta_i\right] = \frac{1}{n^2}\sum_{i=1}^n V\left[\eta_i\right] \le 2n\delta M. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

(4) For n large enough, we know that the inequality $|e_n - e| \le \frac{1}{2}\varepsilon$ holds. Additionally, by (1), (2) and Theorem 2.9, we have

$$\begin{aligned} & Ch\left\{\left|\frac{1}{n}\sum_{i=1}^n \eta_i - e\right| \ge \varepsilon\right\} \\ = \quad & Ch\left\{\left|\frac{1}{n}\sum_{i=1}^n \eta_i - e_n + e_n - e\right| \ge \varepsilon\right\} \\ \le \quad & Ch\left\{\left(\left|\frac{1}{n}\sum_{i=1}^n \eta_i - e_n\right| + |e_n - e|\right) \ge \varepsilon\right\} \end{aligned}$$

$$\leq \quad Ch\left\{\left(\left|\frac{1}{n}\sum_{i=1}^{n}\eta_i - e_n\right|\right) \geq \frac{1}{2}\varepsilon\right\}$$

$$\leq \quad \frac{1}{4\varepsilon^2}V\left[\frac{1}{n}\sum_{i=1}^{n}\eta_i\right] \qquad \cdots\cdots\cdots(3)$$

$$\leq \quad \frac{\delta M}{2\varepsilon^2}.$$

(5) For any $n = 1, 2, \cdots$, any given δ and n large enough, we note that $E\left[|\xi_i - e|\right] \leq +\infty$ such that the inequality

$$Ch\{\varsigma_i \neq 0\} = Ch\{|\xi_i - e| \geq n\delta\}$$

$$= \int_{|\xi - e| \geq n\delta} d\Phi_{\xi_i}(\xi - e)$$

$$\leq \frac{1}{n\delta}\int_{|\xi - e| \geq n\delta} |\xi - e| \, d\Phi_{\xi_i}(\xi - e)$$

$$\leq \frac{\delta}{n}.$$

By Theorem 2.11, we obtain

$$Ch\left\{\sum_{i=1}^{n}\varsigma_i \neq 0\right\} \leq \sum_{i=1}^{n} Ch\{\varsigma_i \neq 0\} \leq \delta. \qquad \cdots\cdots\cdots(4)$$

Hence, for n large enough, by (3), (4) and Lemma 3.10, we have

$$Ch\left\{\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i - e\right| \geq 2\varepsilon\right\}$$

$$= \quad Ch\left\{\left|\frac{1}{n}\sum_{i=1}^{n}\eta_i + \frac{1}{n}\sum_{i=1}^{n}\varsigma_i - e\right| \geq 2\varepsilon\right\}$$

$$\leq \quad Ch\left\{\left(\left|\frac{1}{n}\sum_{i=1}^{n}\eta_i - e\right| + \left|\frac{1}{n}\sum_{i=1}^{n}\varsigma_i\right|\right) \geq 2\varepsilon\right\}$$

$$\leq \quad Ch\left\{\left|\frac{1}{n}\sum_{i=1}^{n}\eta_i - e\right| \geq \varepsilon\right\} + \left(\left|\frac{1}{n}\sum_{i=1}^{n}\varsigma_i\right| \neq 0\right)$$

$$\leq \quad \frac{\delta M}{2\varepsilon^2} + \delta.$$

Note that δ is arbitrary in the definition given above. Then we can have

$$\begin{aligned} 0 &\le \lim_{n\to\infty} Ch\left\{\left|S_n - e\right| \ge \varepsilon\right\} \\ &= \lim_{n\to\infty} Ch\left\{\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i - e\right| \ge \varepsilon\right\} \\ &\le \lim_{\delta\to 0}\left(\frac{\delta M}{2\left(\frac{1}{2}\varepsilon\right)^2} + \delta\right) \\ &= 0 \end{aligned}$$

for any given $\varepsilon > 0$.

The theorem is proved.

4 Conclusion

In this paper, the WLLN for hybrid variables based on chance measure was proved. We would like to emphasize that the result obtained quite similar to the WLLN for random variables based on probability. We take advantage of hybrid variables based on chance measure to develop our theory. In future research, we will be interested in applications of the WLLN for hybrid variables based on chance measure, such as, the analytic hierarchy process with hybrid variables [31], the routing algorithm with hybrid variables [32], the water delivery optimization with hybrid variables [33].

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References

[1] H. Kwakernaak, Fuzzy random variables-I: definitions and theorems, *Information Sciences*, Vol.15, No.1, (1978), pp.1-29.

[2] H. Kwakernaak, Fuzzy random variables-II: Algorithms and examples for the discrete case, *Information Sciences*, Vol.17, No.3, (1979), pp.253-278.

[3] M.L. Puri, D.A. Ralescu, Limit theorems for random compact sets in Banach spaces, *Mathematical Proceedings of the Cambridge Philosophical Society* , Vol.97, (1985), pp.151-158.

[4] M.L. Puri, D.A. Ralescu, Fuzzy random variables, *Journal of Mathematical Analysis and Applications*, Vol.114, No.2, (1986), pp.402-422.

[5] W.E. Stein, K. Talati, Convex fuzzy random variables, *Fuzzy Sets and Systems*, Vol.6, No.3, (1981), pp. 271-283.

[6] R. Kruse, K.D. Meyer, *Statistics with Vague Data*, D. Reidel Publishing Company,Dordrecht, 1987.

[7] Y. Liu, and B. Liu, Fuzzy random variables: A scalar expected value operator, *Fuzzy Optimization and decision making*, Vol.2, No.2, (2003), pp.143-160.

[8] B. Liu, *Theory and Practice of Uncertain Programming*, Physica-Verlag, Heidelberg, 2002.

[9] B. Liu, A survey of credibility theory, *Fuzzy Optimization and decision making*, Vol.5, No.4, (2006), pp.387-408.

[10] C. Zhang, R. Zhao, and W. Tang, Optimal run lengths in deteriorating production processes in random fuzzy environments, *Computers & Industrial Engineering*, Vol.57, No.3, (2009), pp.941-948.

[11] J.Watada, S.Wang, and W. Pedrycz, Building Confidence-Interval-Based Fuzzy Random Regression Models, *IEEE Transactions on Fuzzy Systems*, Vol.17, No.6, (2009), pp.1273-1283.

[12] J. Li and B. Gao, On chance maximization model in fuzzy random decision systems, *Mathematical and Computer Modelling*, Vol.50, No.3-4, (2009), pp.453-464.

[13] W. Huang, L. Ding, B. Wen and B. Gao, Project Scheduling Problem for Software Development with Random Fuzzy Activity Duration Times, *Advances in Neural Networks*, Springer-Verlag Berlin, (2009), pp.60-69.

[14] Y. Feng, Chance-Constrained Goal Programming Models with Random and Fuzzy Parameters, *Proceedings of the Eight International Conference on Information and Management Sciences*, Vol.8, (2009), pp.555-559.

[15] D. Dubois, H. Prade, *Possibility Theory*, Plenum Press, New York, 1988.

[16] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems*, Vol.100, (1999), pp.9-34.

[17] R. Kruse, The strong law of large numbers for fuzzy random variables, *Information Sciences*, Vol.28, No.3, (1982), pp.233-241.

[18] M. Miyakoshi, M. Shimbo, Strong law of large numbers for fuzzy random variables, *Fuzzy Sets and Systems*, Vol.12, No.2, (1984), pp.133-142.

[19] R.L. Taylor, P.Z. Daffer, R.F. Patterson, *Limit Theorems for Sums of Exchangeable Random Variables*, Rowman and Allanheld Publishers.Tozowa, NJ.1985.

[20] E.P. Klement, M.L. Puri, D.A. Ralescu, Limit theorems for fuzzy random variables, *Proceedings of the Royal Society of London*. Series A, Mathematical and Physical Sciences, Vol.407, (1986), pp.171-182.

[21] H. Inoue, A strong law of large numbers for fuzzy random sets, *Fuzzy Sets and Systems*, Vol.41, No.3, (1991), pp.285-291.

[22] Y.K. Kim, A strong law of large numbers for fuzzy random variables, *Fuzzy Sets and Systems*, Vol.111, No.3, (2000), pp.319-323.

[23] S.Y. Joo, Y.K. Kim, Kolmogorov's strong law of large numbers for fuzzy random variables, *Fuzzy Sets and Systems*, Vol.120, No.3, (2001), pp.499-503.

[24] S.Y Joo, Y.K. Kim, J.S. Kwon, On Chung's type law of large numbers for fuzzy random variables, *Statistics and Probability Letters*, Vol.74, No.1, (2005), pp.67-75.

[25] S. Li,Y. Ogura, Strong laws of large numbers for independent fuzzy set-valued random variables, *Fuzzy Sets and Systems*, Vol.157, No.19, (2006), pp.2569-2578.

[26] F.N. Proske, M.L. Puri, Strong Law of Large Numbers for Banach Space Valued Fuzzy Random Variables, *Journal of Theoretical Probability*, Vol.15, No.2, (2002), pp.543-551.

[27] R.L. Taylor, L. Seymour, Y. Chen, Weak Laws of Large Numbers for Fuzzy Random Sets, *Nonlinear Analysis*, Vol.47, No.2, (2001), pp.1245-1256.

[28] Y. Chen, Y. Liu, A strong law of large numbers in credibility theory, *World Journal of Modelling and Simulation*, Vol.2, No.5, (2006), pp.331-337.

[29] B. Liu, *Uncertainty Theory*, 3rd ed., Springer-Verlag, Berlin, 2007.

[30] X. Li, B. Liu, Chance measure for hybrid events with fuzziness and randomness, *Soft Computing-A Fusion of Foundations*, *Methodologies and Applications*, Vol.13, No.2, (2009), pp.105-115.

[31] M. Alias, S. Hashim and S. Samsudin, Using Fuzzy Analytic Hierarchy Process for Southern Johor River Ranking, *International Journal of Advances in Soft Computing and Its Applications*, Vol.1, No.1, (2009), pp.62-76.

[32] S. Upadhyaya and R. Setiya, Ant Colony Optimization: A Modified Version, *International Journal of Advances in Soft Computing and Its Applications*, Vol.1, No.2 (2009), pp.77-90.

[33] R. Bagher and J. Payman, Water Delivery Optimization Program, of Jiroft Dam Irrigation Networks by Using Genetic Algorithm, *International Journal of Advances in Soft Computing and Its Applications*, Vol.1, No.2 (2009), pp.151-161.