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An algorithm for solving fractional differential equations using conformable optimized decomposition method

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Abstract

In this paper, a solution of the linear and nonlinear fractional differential equations is proposed by means of the Optimized Decomposition Method (ODM) formulated with the use of the conformable fractional derivative. This method designs a new optimal construction for the series solutions based on a linear approximation to the nonlinear equation. Thus, the solution in this way gives a high accuracy and closeness to the exact solution. Several numerical results are performed via figures and tables to verify our findings.

***Keywords**:* Conformable fractional derivative; conformable fractional integral; conformable optimized decomposition method; nonlinear differential equations.

1 Introduction

Fractional calculus dates back to the end of the 17th century. Leibniz discussed the L'Hopital derivative of the order 1⁄2. The fractional derivative has many mathematical valuable tools and it has extensive useful applications in many applied sciences fields such as physics and engineering [1-4].

In recent years, the establishment of an accurate or approximate solution to linear or nonlinear ordinary differential equations has been the main goal of many researchers and scientists. One of these methods is the Adomian Decomposition Method (ADM), which was proposed by Adomian for the first time. In particular, G. Adomian introduced in the 1980s a new method to solve nonlinear functional equations [5,6]. As a result, such method had become one of most well-known systematic method that can be used to generate practical solutions for wide equations, including algebraic equations, ordinary and partial differential equations, integral equations and integro-differential equations

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[7,11]. Besides, this method has recently proved its ability in solving linear and nonlinear fractional differential equations.

In the same regard, Z. Odibat [12,13] suggested the so-called Optimized ADM (or simply OADM), as this method relies on the ADM in assuming a specific series for dealing with the nonlinear term. It has been shown that this method can provide more accurate and stronger approximation solutions than other methods. Recently, W. Beghami [14] applied the Laplace transform along with the OADM to study a nonlinear system consisting of fractional partial differential equations via the Caputo sense.

R. Khalil introduced in [15] a new definition of the fractional derivative called the conformable fractional derivative so that such definition can provide the derivatives with more naturality. As consequence, many studies and researches were carried out by a number of researchers, the most famous of whom is M. Abu Hammad, I. Jibril and A. Dababneh, whereby they applied the conformable derivative definition to many statistical distributions, see [17-21]. In this research paper, the OADM is applied to solve some kinds of conformable fractional differential equations. In order to show the efficiency of the proposed method, the generated approximate solutions for under consideration equations are compared with their exact solutions via several numerical results.

This paper is organized as follows: In Section 2, we recall some basic definitions and properties of conformable fractional derivative and fractional integral, which will be used later. In Section 3, we introduce the OADM formulated in the sense of conformable fractional derivative. An algorithm for applying the OADM is then given in Section 4. In Section 5, several numerical tests are shown, i.e. the approximate solutions for several conformable fractional differential equations are compared with their exact solutions, and the error analysis are then discussed. Finally, the conclusion of this paper is summarized in Section 6.

2 Preliminaries and backgrounds

In this section, we aim to give some basic definitions about the conformable fractional derivative and the conformable fractional integral.

Definition 2.1 [15] Given a function $f:[0,\infty) \to \mathbb{R}$, and $t > 0$, $\alpha \in (0,1)$. The conformable derivative of f is defined as

$$\mathcal{D}^{(\alpha)} f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}. \tag{1}$$

If f is α-differentiable on $(0, \alpha)$ for some $\alpha > 0$. If $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then we might define

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t),$$

where $f^{(\alpha)}(t)$ and $\mathcal{D}^{(\alpha)} f(t)$ denote both to the conformable fractional derivative f of order α.

Theorem 2.1 [15] If a function $f: [0,\infty) \to \mathbb{R}$ is α–differentiable at $t_0 > 0$, where $\alpha \in (0,1)$. Then f is continuous at t_0.

Definition 2.2 [16] The conformable fractional integral of the function f is defined by

$$I_\alpha^a(f)(t) = \int_a^t f(x)\, d\alpha\,(x,a) = \int_a^t (x-a)^{\alpha-1} f(x)\, dx.$$

When $\alpha = 0$, we write $d\alpha(x)$. Similarly, in the right case, we have the operator I_α^a are called the conformable fractional integral of order $0 < \alpha \leq 1$.

3 Conformable optimized decomposition method

In this section, the OADM is explained in details to obtain the solutions to some kinds of linear and nonlinear fractional differential equations that have the following general form:

$$\mathcal{D}_t^\alpha \phi(t) + F\big(\phi(t)\big) = g(t), \tag{2}$$

along with the initial condition

$$\phi(0) = h, \phi^\alpha(0) = l, \tag{3}$$

where $t > 0$, $n < \alpha \leq n+1$, h and l are arbitrary constants and $\mathcal{D}_t^\alpha \phi(t)$ is the conformable fractional derivative of $\phi(t)$. Herein, $L = \mathcal{D}_t^\alpha \phi(t)$, F represents a general nonlinear part, whereas $g(t)$ is also denoted to another linear part. Now, by operating the integral operator I^α, which is the inverse of L, to equation (2), and then by using the considered initial condition reported in (3), we get

$$\phi(t) = M(t) + I^\alpha\Big(F\big(\phi(t)\big)\Big), \tag{3}$$

where

$$M(t) = \begin{cases} I^\alpha[g(t)] + \phi(0), if\ 0 < \alpha \leq 1, \\ I^\alpha[g(t)] + \phi(0) + \phi^{(\alpha)}(t)\dfrac{t^\alpha}{\alpha}, if\ 1 < \alpha \leq 2. \end{cases} \tag{4}$$

The conformable optimized decomposition method solution construct as the infinite series:

$$\phi(t) = \sum_{k=0}^{\infty} w_k(t), \tag{5}$$

and the nonlinear terms can be expressed as:

$$F\big(\phi(t)\big) = \sum_{k=0}^{\infty} Q_k(t), \tag{6}$$

where $Q_k(t)$ are called the Adomian polynomials, for $k = 1,2,\cdots$. Actually, these polynomials can be determined from the following relation

$$Q_k(t) = \frac{1}{k!}\frac{d^k}{d\theta^k}\left[F\left(\sum_{i=0}^{k}\theta^i w_i(t)\right)\right]\Bigg|_{\theta=0}. \quad (7)$$

Now, we apply the main idea of OADM, which can be carried out by creating a linear approximation of the corresponding nonlinear function

$$M(t) = \phi(t) + I^\alpha\left(F\big(\phi(t)\big)\right), \quad (8)$$

that can be linearized by a first-order Taylor series expansion at $t = 0$ as follow

$$H(t) = F\big(\phi(t)\big) + C_0 t, \quad (9)$$

where $C_0 = \frac{\partial F}{\partial \phi}\Big|_{t=0}$. In this regard, the component functions $\{w_k(t)\}_{k=0}^{\infty}$ can be determined recursively by the following relations

$$\begin{cases} w_0(t) = M(t), \\ w_1(t) = I^\alpha[Q_0(t)], \\ w_2(t) = I^\alpha[Q_1(t) + C_0 w_1(t)], \\ w_{k+1}(t) = I^\alpha\big[Q_k(t) + C_0\big(w_k(t) - w_{k-1}(t)\big)\big], \qquad k \geq 2, \end{cases} \quad (10)$$

such that $F(\sum_{k=0}^{\infty} w_k(t)) = \sum_{k=0}^{\infty} Q_k(t)$. It can be clearly seen that if the decomposition series $\sum_{k=0}^{\infty} w_k(t)$ converges, then $\lim_{k\to\infty} w_k = 0$.

4 Algorithm for OADM

For finding the approximate solution to the fractional differential equation, there are five steps that need to be considered. We state them below for completeness.

Input. The iteration k, initial conditions $\phi(0) = h, \phi^\alpha(0) = l$, the nonlinear term F.
Output. An approximate solution of the fractional differential equation.
Steps. There are five steps that should be considered. These steps are:

- Step 1. Operate I^α to equation (2).
- Step 2. Find the Adomian polynomials by the following formula:

$$Q_k(t) = \frac{1}{k!}\frac{d^k}{d\theta^k}\left[F\left(\sum_{i=0}^{k}\theta^i w_i(t)\right)\right]\Bigg|_{\theta=0}.$$

- Step 3. Solve $C_0 = \frac{\partial F}{\partial \phi}\Big|_{t=0}$.
- Step 4. Solve the following recursive states:

$$\begin{cases} w_0(t) = M(t), \\ w_1(t) = I^{\alpha}[Q_0(t)], \\ w_2(t) = I^{\alpha}[Q_1(t) + C_0 w_1(t)], \\ w_{k+1}(t) = I^{\alpha}\big[Q_k(t) + C_0\big(w_k(t) - w_{k-1}(t)\big)\big], k \geq 2. \end{cases}$$

- Step 5. Find the approximate solution from $w_0(t)$ to $w_{k+1}(t)$ of the fractional differential equation at hand.

5 Representation of exact and numerical solutions

This section illustrates the efficiency of the OADM formulated in the sense of conformable fractional derivative and presents approximate solutions for some linear and nonlinear fractional differential equations. In this regard, the proposed algorithm provided in the previous section is used through the given examples via a prepared computer code (Mathematica 13).

Example 5.1: Consider the following linear fractional differential equation

$$y^{\alpha} - y = 0,$$

with the initial condition

$$y(0) = 1, ,$$

where $t > 0$ and $0 < \alpha \leq 1$. The exact solution of this equation is given by

$$y(t) = e^{\frac{t^{\alpha}}{\alpha}}.$$

In order to apply on our proposed scheme, we let $\alpha = 1$ and $y^{\alpha} = y$. Consequently, the conformable optimized decomposition method solutions are:

$$y(t) = \sum_{k=0}^{\infty} w_k(t),$$

where $C_0 = 1$. The suggested solution here is $y(t) = \sum_{k=0}^{\infty} w_k(t)$, where the components $\{w_k(t)\}_{k=0}^{\infty}$ can be evaluated by the following recurrence relations

$$\begin{cases} w_0(t) = y_0\ , \\ w_1(t) = \int_0^t -w_0\ t^{\alpha-1}\, dt\ , \\ w_2(t) = \int_0^t (-w_1 + C_0 w_1)\ t^{\alpha-1}\, dt\ , \\ w_{k+1}(t) = \int_0^t \Big(Q_k(t) + C_0\big(w_k(t) - w_{k-1}(t)\big)\Big)\ t^{\alpha-1}\, dt\ , \qquad k \geq 2, \end{cases}$$

where $Q_k(t)$ is defined by

$$Q_k(t) = \frac{1}{k!}\frac{d^k}{d\theta^k}\left[\left(w_0(t) + \theta w_1(t) + \theta^2 w_2(t) + \theta^3 w_3(t) + \cdots +\right)\right]\Big|_{\theta=0}.$$

In this example, we only use the first ten terms to approximate the exact solution. However, from the error columns shown in Table 1, one can see that the absolute values of these error are very small. This would lead us to conclude that the conformable optimized decomposition method has a high convergence order. For further explanation, Figure 1 represents the graphs of the exact solution and the approximate solution of the considered problem at different values of α.

Table 1: Absolut Errors for different values of α of Example 5.1

t	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.7$
0.1	1.1×10^{-16}	1.1×10^{-16}	5.5×10^{-16}	2.4×10^{-14}
0.2	5.5×10^{-16}	9.4×10^{-15}	2.0×10^{-13}	5.0×10^{-12}
0.3	4.3×10^{-14}	5.1×10^{-13}	7.0×10^{-12}	1.1×10^{-10}
0.4	1.0×10^{-12}	8.8×10^{-12}	8.7×10^{-11}	1.0×10^{-9}
0.5	1.1×10^{-11}	7.9×10^{-11}	6.1×10^{-10}	5.6×10^{-9}
0.6	8.6×10^{-11}	4.7×10^{-10}	3.0×10^{-9}	2.2×10^{-8}
0.7	4.6×10^{-10}	2.1×10^{-9}	1.1×10^{-8}	7.4×10^{-8}
0.8	2.0×10^{-9}	8.1×10^{-9}	3.7×10^{-8}	2.0×10^{-7}
0.9	7.3×10^{-9}	2.5×10^{-8}	1.0×10^{-7}	5.0×10^{-7}
1	2.3×10^{-8}	7.3×10^{-8}	2.6×10^{-7}	1.1×10^{-6}

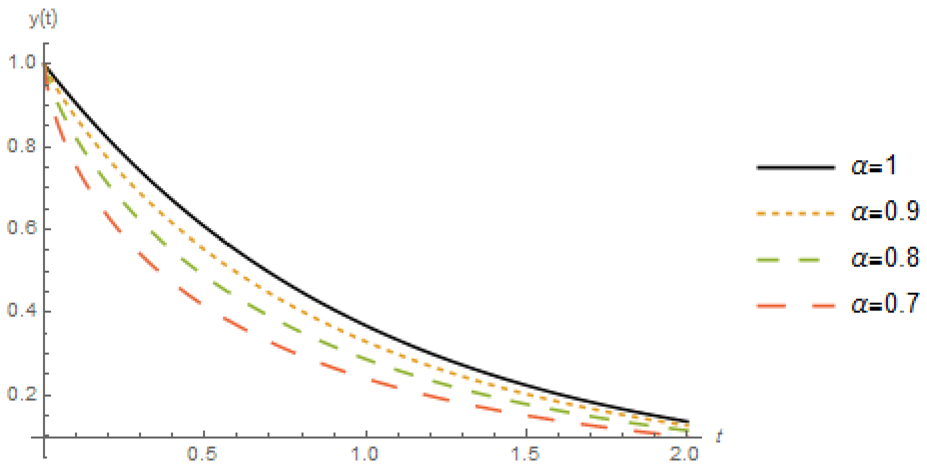


Figure 1. Approximate solution of $y(t)$ for different values of α for Example 5.1

Example 5.2: Consider the following nonlinear fractional differential equation:

$$y^{\alpha} - 1 + y^2 = 0,$$

subject to the initial condition

$$y(0) = -\frac{1}{2}$$

where $t > 0$ and $0 < \alpha \leq 1$. Herein, the exact solution of the above equation is given by

$$y(t) = \frac{\tanh(t) + y_0}{y_0 \tanh(t) + 1}.$$

To apply on the proposed method, we let $\alpha = 1$ and $y^\alpha = 1 - y^2$, where $C_0 = -1$. Then the conformable optimized decomposition method suggests the solution $y(t) = \sum_{n=0}^{\infty} w_n(t)$, where

$$\begin{cases} w_0(t) = y_0\,, \\ w_1(t) = \int_0^t t^{\alpha-1}\,dt - \int_0^t (w_0)^2 t^{\alpha-1}\,dt\,, \\ w_2(t) = \int_0^t (-2w_1 w_0 + C_0 w_1)\, t^{\alpha-1}\,dt\,, \\ w_{k+1}(t) = \int_0^t \Big(Q_k(t) + C_0\big(w_k(t) - w_{k-1}(t)\big)\Big)\, t^{\alpha-1}\,dt\,, k \geq 2\,. \end{cases}$$

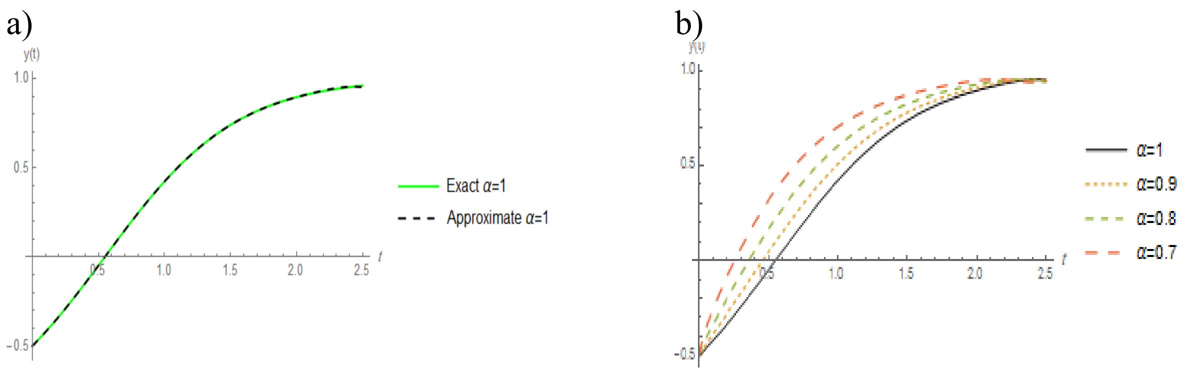
Here the polynomial $Q_k(t)$ is defined by

$$Q_k(t) = \frac{1}{k!}\frac{d^k}{d\theta^k}\Big[\big((w_0(t) + \theta w_1(t) + \theta^2 w_2(t) + \theta^3 w_3(t) + \cdots +)\big)^2\Big]\Big|_{\theta=0}\,,$$

Table 2 reported below describes the exact solution and the approximate solution regarding Example 5.2 for $\alpha = 1$. Furthermore, Figure 2 represents the graph of the exact solution and approximate solutions for different values of α.

Table 2: Numerical results for u_1 of Example **5.2**

t	Exact	Approximate solutions				Absolut error
	$\alpha = 1$	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.8$	$\alpha = 0.7$	$\alpha = 1$
0.1	−0.4213	−0.4213	−0.3879	−0.3374	−0.2582	0
0.2	−0.3357	−0.3357	−0.2805	−0.2015	−0.0860	3.1×10^{-15}
0.3	−0.2442	−0.2442	−0.1716	−0.0720	0.0656	4.9×10^{-14}
0.4	−0.1482	−0.1482	−0.0621	0.0512	0.2001	1.0×10^{-11}
0.5	−0.0492	−0.0492	0.0460	0.1670	0.3185	1.4×10^{-10}
0.6	0.0506	0.0506	0.1511	0.2741	0.4217	3.1×10^{-10}
0.7	0.1495	0.1495	0.2512	0.3716	0.5106	4.1×10^{-9}
0.8	0.2455	0.2455	0.3448	0.4592	0.5867	3.5×10^{-8}
0.9	0.3369	0.3369	0.4311	0.5368	0.6513	1.2×10^{-7}
1	0.4224	0.4224	0.5093	0.6048	0.7060	1.7×10^{-7}



c)

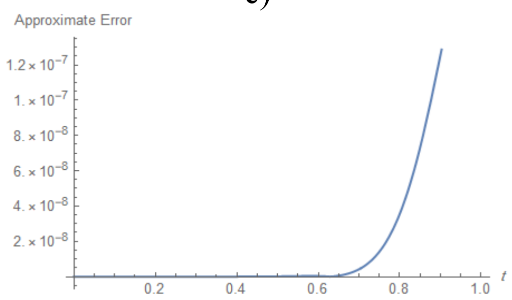


Figure 2. (a) Exact and approximate solutions $y(t)$, (b) Approximate solution of of $y(t)$ for different values of α (c) Absolute Error of $y(t)$ when $\alpha = 1$ for Example 5.2

Example 5.3: Consider the following nonlinear fractional differential equation [18]:

$$y^{\alpha} + \pi^2 e^{-y} = 0,$$

subject to the initial conditions

$$y(0) = 0, \qquad y^{(\alpha)}(0) = \pi.$$

where $t > 0$ and $1 < \alpha \leq 2$. The exact solution of the above problem is given by

$$y(t) = \ln\left(1 + \sin\left(\frac{\pi}{\alpha} t^{\alpha}\right)\right),$$

Herein, we let $\alpha = 2$ and $y^{2\alpha} = -\pi^2 e^{-y}$, where $C_0 = -\pi^2$. Now, since the conformable optimized decomposition method suggests the solution $y(t) = \sum_{n=0}^{\infty} w_n(t)$, we get

$$\begin{cases} w_0(t) = y_0 + y_1 \dfrac{t^{\alpha}}{\alpha}, \\ w_1(t) = \displaystyle\int_0^t \left(\int_0^t Q_0 t^{\alpha-1}\, dt\right) t^{\alpha-1}\, dt \ , \\ w_2(t) = \displaystyle\int_0^t \left(\int_0^t (Q_1 + C_0 w_1) t^{\alpha-1}\, dt\right) t^{\alpha-1}\, dt \\ w_{k+1}(t) = \displaystyle\int_0^t \left(\int_0^t \Big(Q_k + C_0\big(w_k(t) - w_{k-1}(t)\big)\Big) t^{\alpha-1}\, dt\right) t^{\alpha-1}\, dt. \end{cases}$$

Here the polynomial $Q_k(t)$ is defined by

$$Q_k(t) = \frac{1}{k!}\frac{d^k}{d\theta^k}\left[\left(\left(w_0(t) + \theta w_1(t) + \theta^2 w_2(t) + \theta^3 w_3(t) + \cdots +\right)\right)^2\right]\Bigg|_{\theta=0}.$$

Table 3 displays the error of the exact and approximate solutions at $\alpha = 2$. In this example, the numerical results demonstrate high precision. Figure 3 depicts the graph of the exact and approximate solutions of the problem at hand for various values of α.

Table 3: Absolut errors for different values of α of Example 5.3

t	$\alpha = 2$	$\alpha = 1.9$	$\alpha = 1.7$	$\alpha = 1.5$	$\alpha = 1.2$
0.1	1.2×10^{-9}	1.3×10^{-9}	2.0×10^{-11}	1.0×10^{-11}	1.0×10^{-10}
0.2	1.0×10^{-9}	1.5×10^{-9}	1.1×10^{-11}	3.0×10^{-10}	8.6×10^{-10}
0.3	3.7×10^{-10}	3.0×10^{-9}	9.4×10^{-12}	9.1×10^{-10}	2.0×10^{-10}
0.4	2.5×10^{-11}	1.4×10^{-9}	9.5×10^{-12}	1.0×10^{-10}	1.9×10^{-9}
0.5	1.0×10^{-10}	1.0×10^{-10}	1.5×10^{-9}	6.6×10^{-10}	1.3×10^{-9}
0.6	2.6×10^{-9}	1.2×10^{-10}	4.3×10^{-8}	1.5×10^{-8}	2.5×10^{-8}
0.7	2.8×10^{-10}	1.7×10^{-9}	7.5×10^{-7}	3.6×10^{-7}	7.9×10^{-6}
0.8	1.9×10^{-8}	6.5×10^{-8}	8.9×10^{-6}	5.3×10^{-6}	2.2×10^{-6}
0.9	6.0×10^{-7}	1.3×10^{-6}	7.7×10^{-5}	5.3×10^{-5}	9.6×10^{-5}
1	9.9×10^{-6}	1.9×10^{-5}	5.1×10^{-5}	3.8×10^{-5}	5.4×10^{-5}

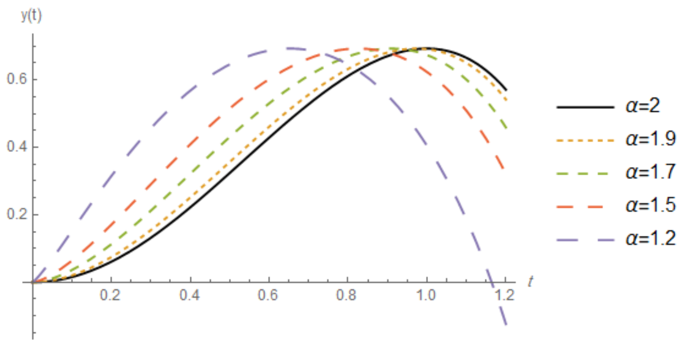


Figure 3. Approximate solution of $y(t)$ for different values of α for Example 5.3

6 Conclusion

In this paper, the so-called conformable optimized decomposition method has been used to study the approximate solution of the fractional differential equations. The approximate solution has been obtained based on the linear approximation for nonlinear problems. The behavior of the proposed method has been shown by solving and studying some numerical examples. Based on the gained numerical results, we can conclude that the proposed method is effective and it is sufficiently suitable for nonlinear problems.

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