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Distance-related Properties of Corona of Certain Graphs

S. Sriram[1], Atulya K Nagar[2], and K.G. Subramanian[2]

[1]VELS University,
Pallavaram, Chennai 600 117 India
e-mail: sriram.discrete@gmail.com
[2]Department of Mathematics and Computer Science,
Liverpool Hope University, Liverpool L16 9JD UK
e-mail:nagara@hope.ac.uk, kgsmani1948@gmail.com

Abstract

A graph G is called a m-eccentric point graph if each point of G has exactly m ≥ 1 eccentric points. When m=1, G is called a unique eccentric point (u.e.p.) graph. Using the notion of corona of graphs, we show that there exists a m-eccentric point graph for every m ≥ 1 . Also, the eccentric graph G_e of a graph G is a graph with the same points as those of G and in which two points u and v are adjacent if and only if either u is an eccentric point of v or v is an eccentric point of u in G. We obtain the structure of the eccentric graph of corona G°H of self-centered or non-self-centered u.e.p graph G with any other graph H and obtain its domination number.

Keywords: *Domination, Eccentricity, Eccentric Graph.*

1 Introduction

The notion of distance [2] in graphs has been studied in the context of many applications such as communication networks. The distance related parameter, known as eccentricity of a point in a graph and the associated notions of eccentric points, m-eccentric point graphs[1, 3, 6] and in particular, unique eccentric point (*u.e.p*)graphs [4], have also been well investigated. Another kind of graph known as corona [6] G°H of two graphs G and H has also been well studied. Also, the concept of eccentric graph G_e of a graph G was introduced in [5], based on the notion of distance among points in G. Here we show, using the notion of corona of graphs, that there exists a m-eccentric point graph for every m $\geq$ 1. We also obtain the eccentric graph of corona G°H where H is any graph and G is either self-centered *u.e.p* graph or non-self-centered *u.e.p* graph and obtain its domination number.

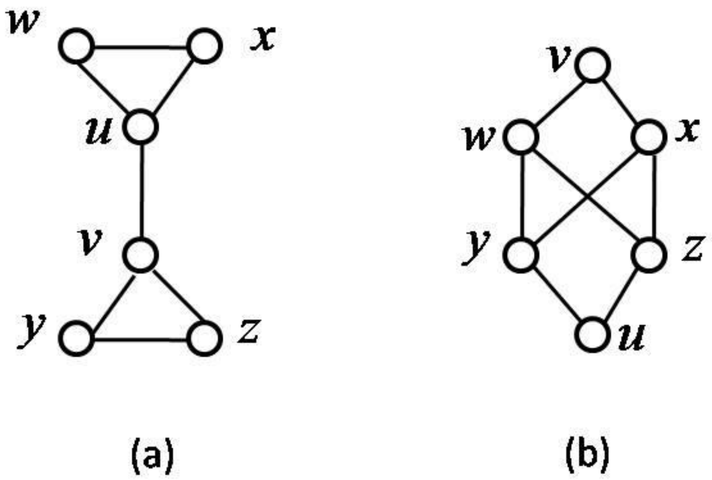

Fig. 1: a) Graph G b) Graph G_e

We recall here certain basic definitions {1, 3, 6] related to graphs. A graph

G = (V, E) consists of a finite non-empty set V (also denoted by V(G)) whose elements are called points or vertices and another set E (or E(G)) of unordered pairs of distinct elements of V, called edges. In a graph G, the distance $d_G(u,v)$ or d(u,v) , when G is understood, between two points u and v is the length of the shortest path between u and v. The eccentricity $e_G(u)$ or simply, e(u) of a point u in G is defined as $e(u) = \max_{v \in V(G)} d(u,v)$. For two points u, v in G, the point v is an eccentric point of u if d(u,v) = e(u). We denote by E (v), the set of all eccentric points of a point v in G. A graph G is called a m-eccentric point graph if |E(u)|, the number of elements of E(u) equals m, for all u in V(G). When m=1, G is called a unique eccentric point (*u.e.p*) graph. The radius r(G) and the diameter diam(G) of a graph G are respectively defined as r(G) = min {e(u) / for all u $\in$ V(G)} and diam(G) = max {e(u) / for all u $\in$ V(G)}. A graph G is called a self-centered graph if r(G) = diam(G).

The eccentric graph [5] G_e of a graph G is a graph with the same points as those of G and in which two points u and v are adjacent if and only if either u is an eccentric point of v or v is an eccentric point of u in G. A graph G and its eccentric graph G_e are shown in Fig.1.

The corona {6] G°H of two graphs G and H is a graph made of one copy of G with points $v_1, v_2, v_3, ..., v_n$, $n \geq 1$ and n copies of another graph H such that for every i, 1 ≤ i ≤ n , the point v_i is joined with all the points of the i th copy of H.

We also need the following well-known notions. A complete graph K_n on n points, is a graph in which there is an edge between every pair of distinct points. The complement $\overline{G}$ of a graph G is a graph having the same points as those of G and such that two points x and y are adjacent in $\overline{G}$ if and only if x and y are not adjacent in G. The union $G_1 \cup G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ and the join $G_1 + G_2$ of G_1 and G_2 is a graph obtained from $G_1 \cup G_2$ by joining every point of $G_1$with every other point of G_2. For three or more graphs $G_1, G_2, G_3, ..., G_n$ the sequential join $G_1 + G_2 + G_3 + ... + G_n$ is the graph $(G_1 + G_2) \cup (G_2 + G_3) \cup ... \cup (G_{n-1} + G_n)$. In a graph G=(V,E) a subset S ⊂ V is called a dominating set if each point u in V-S has a neighbour in S i.e. u is adjacent to some point in S. The cardinality of a minimum dominating set of a graph G is called its domination number and it is denoted by γ(G).

2 Eccentric Point Graphs

In this section we make use of the notions of corona and *u.e.p* graphs to show there exists, for every m ≥1, an m-eccentric point graph.

Lemma 2.1 *Let G be a graph whose eccentric points are* $p_1, p_2, p_3, \dots, p_l$, *for some* $l \geq 1$. *Let H be any other graph. In the corona G°H, all the points of G°H each of which is joined with* p_i , *for some i,* $1 \leq i \leq l$, *are the only eccentric points of G°H.*

Proof. Let $V(G) = \{v_1, v_2, v_3, \dots, v_{n-1}, p_1, p_2, p_3, \dots, p_l\}$ such that p_i's are eccentric points of G. Let H be any graph on m points. Let $V(G^\circ H) = \{v_i, p_k, u_j^t / 1 \leq i \leq n-1, 1 \leq k \leq l, 1 \leq t \leq n \text{ and } 1 \leq j \leq m\}$ such that for a fixed i, with 1 ≤ i ≤ n-*l*, the points u_j^i for all 1≤ j ≤ m, are joined with v_i while for a fixed i, with n-*l* ≤ i ≤ n, the points u_j^i for all 1≤ j ≤ m, are joined with p_i.

Then, let us prove that every point u_j^i for all n-*l*+1 ≤ i ≤ n and 1≤ j ≤ m is an eccentric point. Suppose that for some n-*l*+1 ≤ i ≤ n and 1≤ j ≤ m,

u_j^i is not an eccentric point. Then consider the point v ∈ V(G) whose eccentric point in G is p_i to which the point u_j^i is attached in G°H. Let for some

n-*l*+1 ≤ i ≤ n and 1 ≤ j ≤ m, u_j^k be the eccentric point of v in G°H. Then $e_{G^\circ H}(v) = d_{G^\circ H.}(v, u_j^k) = d_G(v, p_k) + 1 < d_G(v, p_i) + 1 = d_{G^\circ H.}(v, u_j^i)$.

That is $e_{G^\circ H}(v) < d_{G^\circ H.}\left(v, u_j^i\right)$, which is a contradiction and hence every point u_j^i, n-l+1 $\leq$ i $\leq$ n and 1$\leq$ j $\leq$ m is an eccentric point. Now, it remains to prove that (1) no point of G as a point of G°H is an eccentric point in G°H and (2) no point of u_j^i for 1 $\leq$ i $\leq n - l$ and 1$\leq j \leq m$ is an eccentric point in G°H.

In order to prove (1), suppose that u is an eccentric point of G°H. Then there exists a point v $\in$ G°H for which u is the eccentric point. Then u cannot be any v_i, 1$\leq$ i $\leq$ n-l or any p_i, n-l+1 $\leq$ i $\leq$ n for otherwise $e_{G^\circ H}(v) = d_{G^\circ H.}(u, v) < d_{G^\circ H.}(u, v) + 1 = d_{G^\circ H.}\left(v, u_j^i\right)$ which is a contradiction, due to the fact that any path between v and u_j^i passes through either v_i or p_i. Thus no point of G as a point of G°H, can be an eccentric point of G°H.

For proving (2), suppose that u_j^i for some1$\leq$ i $\leq n - l$,1 $\leq j \leq m$ is an eccentric point of some point v in G°H, then $e_{G^\circ H}(v) = d_{G^\circ H.}\left(v, u_j^i\right) = d_{G^\circ H.}(v, v_i) + 1 < d_{G^\circ H.}\left(v, u_j^k\right)$. That is $e_{G^\circ H}(v) < d_{G^\circ H.}\left(v, u_j^k\right)$ for some n-l+1 $\leq$ k $\leq$ n, which is a contradiction.

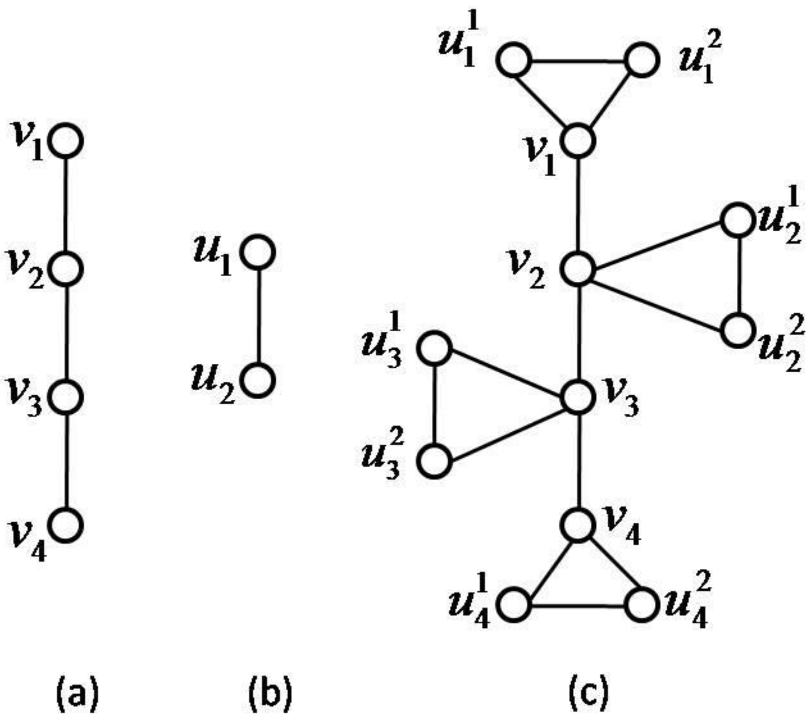

Fig. 2: a) Graph G b) Graph H c) Corona G°H

Remark 2.2. With the graphs G and H as shown in Fig. 2, the eccentric points of G°H are $u_1^1, u_1^2, u_4^1, u_4^2$. It can be noticed that no point v_i, 1$\leq$ i $\leq$ 4, of G is an eccentric point of G°H and all the points of G°H that are joined with the points that are eccentric points of G are the only eccentric points of G°H.

Theorem 2.3. Let G be a *u.e.p* graph on n points and H be any graph on m points. Then the corona of G and H, G°H is a m-eccentric point graph.

Proof. Let G be a *u.e.p* graph with n points $v_1, v_2, v_3, \dots, v_n$ and H be any graph on m points. Let the points of G°H that are points in the k[th] copy of H, be $u_j^k, 1 \leq j \leq m$. For any two points v_i, v_k of G such that v_k is the only eccentric point in G of v_i, by Lemma 2.1, the points u_j^k, $1 \leq k \leq m$, are the eccentric points in G°H of v_i as well as $u_{j,}^i$ $1 \leq j \leq m$. No other point $u_{j,}^r$ for some $r \neq k$, $1 \leq r \leq n$, can be an eccentric point in G°H of v_i or $u_{j,}^i$ $1 \leq j \leq m$, since $d_{G^\circ H}(v_i, u_j^r) = d_G(v_i, v_r) + 1 < d_G(v_i, v_k) + 1 = d_{G^\circ H}(v_i, u_j^k)$ and $d_{G^\circ H}(u_j^i, u_j^r) = d_{G^\circ H}(v_i, u_j^r) + 1 < d_{G^\circ H}(v_i, u_j^k) + 1 = d_{G^\circ H}(u_j^i, u_j^k)$. This implies that $E(v_i) = \{u_1^k, u_2^k, u_3^k, \dots, u_m^k\}$. If v_k is an eccentric point of v_i in G and $E(u_j^p) = \{u_1^q, u_2^q, u_3^q, \dots, u_m^q\}$ if v_q is an eccentric point of v_p in G. Therefore, $|E(u)| = m$, for all points u in G°H and so G°H is a m-eccentric point graph.

As a consequence of the Theorem 2.3, we obtain the following corollary.

Corollary 2.4. For every $m \geq 1$, there exists a m-eccentric point graph.

3 Eccentric Graph of Corona of *u.e.p* Graph with any other Graph

In this section we obtain the eccentric graph of corona of a *u.e.p* graph with any other graph.

Theorem 3.1. Let G be a self-centered *u.e.p* graph on 2n points and H be a graph on m points. Then the eccentric graph $(G°H)_e$ is the union of n copies of $K_1 + \overline{K_m} + \overline{K_m} + K_1$.

Proof. Let G be a self-centered *u.e.p* graph on 2n points and H be a graph on m points. Let $V(G) = \{v_1, v_2, v_3, \dots, v_{2n}\}$ such that v_i and v_{i+n} $(1 \leq i \leq n)$ are eccentric points of each other, in the graph G. Then by Lemma 2.1, all the points u_j^i $(1 \leq i \leq 2n; 1 \leq j \leq m)$ are eccentric points in G°H because all the points of G are eccentric points in G. This implies that the eccentric points of u_j^i and v_i are u_j^{i+n} $(1 \leq i \leq n, 1 \leq j \leq m)$ and the eccentric points of u_j^{i+n} and v_{i+n} are u_j^i $(1 \leq i \leq n, 1 \leq j \leq m)$. Now in $(G°H)_e$, which has the same point set as G°H, the point v_i is adjacent with all the points u_j^{i+n}, each of the points u_j^{i+n} is adjacent with every point u_j^i and all the points u_j^i are adjacent with v_{i+n} ($1 \le i \le n$, $1 \le j \le m$). Therefore, $(G°H)_e$ is the union of n copies of $K_1 + \overline{K_m} + \overline{K_m} + K_1$.

Theorem 3.2. Let H be a graph on m points and G be a non-self-centered *u.e.p* graph on n points having the properties (i) P(G) =EP(G), (ii) |P(G)| =2t, t>1, (iii) for every u in P(G) there is at least one v in V(G) – P(G) such that E(v) ={u}, then $(G°H)_e$ is a union of t copies of $\overline{K_{t_i}} + \overline{K_m} + \overline{K_m} + \overline{K_{t_j}}$, for some $t_i \ge 1$ and $t_j \ge 1$, t_i and t_j depending on G and H.

Proof. Let H be a graph on m points and G be a non-self-centered *u.e.p* graph on n points having the properties (i) P(G) =EP(G), (ii) |P(G)| =2t, t>1, (iii) for every u in P(G) there is at least one v in V(G) – P(G) such that E(v) ={u}.

Let V(G) $=\{v_1, v_2, v_3, \dots, v_n\}$. Let $v_1, v_2, v_3, \dots, v_{2t}$ for some $t \ge 1$ be the peripheral vertices of G, so that |P(G)| =2t. For $1 \le i \le t$, let v_i and v_{i+t} be the eccentric points of each other. Let V(G°H) = V(G) $\cup$ $\{u_{j,}^i\ 1 \le j \le m,\ 1 \le i \le n\}$. Then by Lemma 2.1, all the points u_j^i, $1 \le i \le 2t$, $1 \le j \le m$ are the eccentric points of G°H because $v_1, v_2, v_3, \dots, v_{2t}$ are the eccentric points in G. This implies that for $1 \le i \le t$, $1 \le j \le m$, u_j^{i+t} is the eccentric point of u_j^i, v_i, v_k as well as u_j^k with E(v_k) ={v_i} for v_k ε V(G). Also, $1 \le i \le t$, $1 \le j \le m$, u_j^i is the eccentric point of u_j^{i+t}, v_{i+t}, v_k as well as u_j^k. Since the eccentric graph G_e, of any graph G, is constructed with the same points as those of G and each edge of G_e joins a point x with the eccentric points of x treated as a point of G. Thus, the structure of $(G°H)_e$ is clearly, union of t copies of $\overline{K_{t_i}} + \overline{K_m} + \overline{K_m} + \overline{K_{t_j}}$, for some $t_i \ge 1$ and $t_j \ge 1$. Note that t_i and t_j depend on G and H.

Example 3.3. A non-self-centered *u.e.p* graph G and H on m=2 points are shown in Fig.3. It is clear that in G, the eccentric points are v_1, v_2, v_3 *and* v_4 and E(v_4) = E(v_{10}) = E(v_{12}) = {v_1}; E(v_1) = E(v_5) = E(v_7) = {v_4};

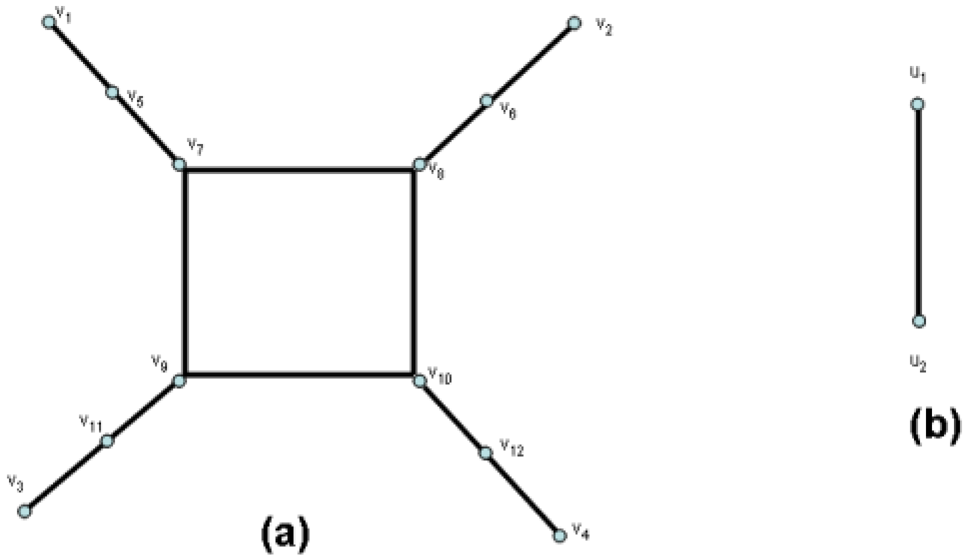

Fig. 3: a) A non-self centered *u.e.p* Graph G b) Graph H

$E(v_2) = E(v_6) = E(v_8) = \{v_3\}$; $E(v_3) = E(v_{11}) = \{v_2\}$. The corona of G and H is shown in Fig.4. Note that $v_1^1, v_2^1, v_1^2, v_2^2, v_1^3, v_2^3, v_1^4, v_2^4$ are the eccentric vertices of $(G°H)_e$. The eccentric graph, $(G°H)_e$ is union of 2 copies of $\overline{K_{t_i}} + \overline{K_m} + \overline{K_m} + \overline{K_{t_j}}$, where m=2; t_i =7 and t_j =7 and it is shown in the Fig.5.

Theorem 3.4. Let G be a self-centered *u.e.p* graph on 2n points and H be a graph on m points. Then the domination number $\gamma(G°H)_e = 2n$.

Proof. Let G be self-centered *u.e.p* graph on 2n points and H be a graph on m points. Now, by Theorem 3.1, $(G°H)_e$ is union of n copies of $K_1 + \overline{K_m} + \overline{K_m} + K_1$. In each copy, there are two v_i's dominating the remaining points in that copy. Therefore, $\gamma(G°H)_e = 2n$.

Theorem 3.5. Let H be a graph on m points and G be a non-self-centered *u.e.p* graph on n points having the properties (i) P(G) =EP(G), (ii) |P(G)| =2t, t>1, (iii) for every u in P(G) there is at least one v in V(G) – P(G) such that E(v) ={u}, then the domination number $\gamma(G°H)_e = 2t$.

Proof. Let H be a graph on m points and G be a non-self-centered *u.e.p* graph on n points having the properties (i) P(G) =EP(G), (ii) |P(G)| =2t, t>1, (iii) for every u in P(G) there is at least one v in V(G) – P(G) such that E(v) ={u}. Then by

Theorem 3.2, $(G°H)_e$ is union of t copies of $\overline{K_{t_i}} + \overline{K_m} + \overline{K_m} + \overline{K_{t_j}}$, for some $t_i \geq 1$ and $t_j \geq 1$, t_i and t_j depending on G and H. In each copy, there are two points dominating the remaining points in that copy. Therefore, $\gamma(G°H)_e = 2t$.

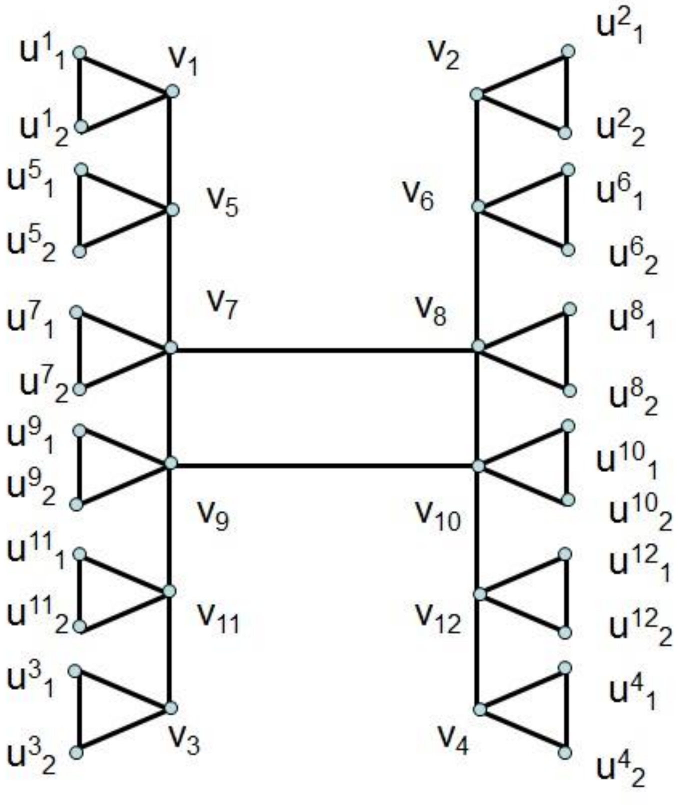

Fig. 4: Corona of G and H, G°H

4 Conclusion

The structure of eccentric graph of m-eccentric point graph can be investigated. Also the problem of finding a graph whose eccentric graph is a m-eccentric point graph remains open.

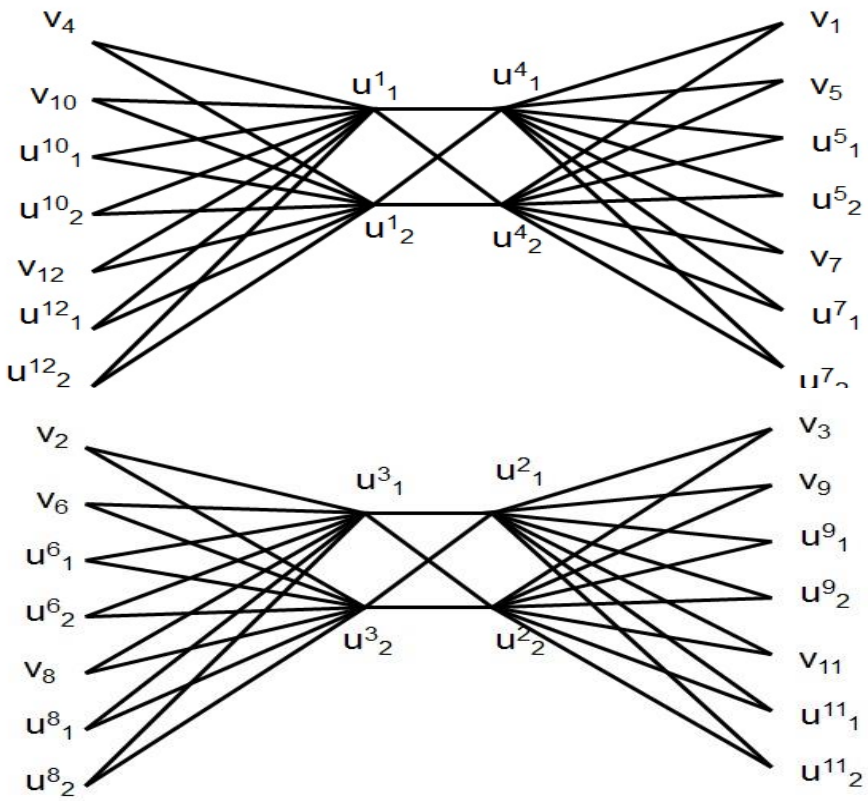

Fig. 5: Eccentric Graph of G°H, $(G°H)_e$

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