Int. J. Advance Soft Compu. Appl, Vol. 8, No. 2, July 2016
ISSN 2074-8523

Secure Domination in Graphs

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Abstract

Let $G=(V,E)$ be a graph. A subset S of V is a dominating set of G if every vertex in $V \setminus S$ is adjacent to a vertex in S. A dominating set S is called a secure dominating set if for each $v \in V \setminus S$ there exists $u \in S$ such that v is adjacent to u and $S_1=(S-\{u\}) \cup \{v\}$ is a dominating set. In this paper we introduce the concept of secure irredundant set and obtain an inequality chain of four parameters.

Keywords: *Secure domination, Secure irredundance, Secure domination number, Upper secure domination number, Secure irredundance number, Upper secure irredundance number.*

1 Introduction

By a graph $G=(V,E)$, we mean a finite, undirected graph with neither loops nor multiple edges. The order $|V|$ and the size $|E|$ of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [4].

The open neighborhood of a vertex $v \in V$ is given by $N(v)=\{u \in V: uv \in E\}$ and its closed neighborhood is $N[v]=N(v) \cup \{v\}$. Given $S \subseteq V$ and $v \in S$, a vertex $u \in V$ is an S-private neighbor of v if $N(u) \cap S=\{v\}$. The set of all S-private neighbors of v is denoted by $PN(v,S)$. If further $u \in V \setminus S$, then u is called an S-external private neighbor (abbreviated S-epn) of v. The set of all S-epns of v is denoted by $EPN(v,S)$. A set $S \subseteq V$ is called a dominating set of G if every vertex in $V \setminus S$ is adjacent to a vertex in S. A dominating set S is called a minimal dominating set if $S \setminus \{v\}$ is not a dominating set for all $v \in S$. The minimum (maximum) cardinality of a minimal dominating set of G is called the domination number (upper domination number) of G and is denoted by $\gamma(G)$ ($\Gamma(G)$). A subset S of V is called an irredundant set if every vertex $v \in S$ has at least one private neighbor. The minimum cardinality of a maximal irredundant set is called irredundance number of G and is denoted by $ir(G)$. The maximum cardinality of a maximal irredundant set in G is called the upper irredundance number of G and is denoted by $IR(G)$. A subset S of V is called an independent set if no two vertices in S are adjacent. The minimum cardinality of a maximal independent set is called the independent domination number of G and is denoted by $i(G)$. The maximum cardinality of a maximal independent set is called the independence number of G and is denoted by $\beta_0(G)$. The following inequality chain was first given in Cockayne et al. [8].

Theorem 1.1. [8] *For any graph G have*

$$Ir(G) \le \gamma(G) \le i(G) \le \beta_0(G) \le \Gamma(G) \le IR(G).$$

This inequality chain is one of the strongest focal points for research in domination theory.

Strategies for protection of a graph $G=(V,E)$ by placing one or more guards at every vertex of a subset S of V, where a guard at v can protect any vertex in its closed neighborhood have resulted in the study of several concepts such as Roman domination, weak Roman domination and secure domination. The concept of secure domination is motivated by the following situation. Given a graph $G=(V,E)$ we wish to place one guard at each vertex of a subset S of V in such a way that S is a dominating set of G and if a guard at v moves along an edge to protect an unguarded vertex u, then the new configuration of guards also forms a dominating set. In other words, for each $u \in V \setminus S$ there exists $v \in S$ such that v is adjacent to u and $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. In this case we say that u is S-defended by v or v S-defends u. A dominating set S in which every vertex in $V \setminus S$ is S-defended by a vertex in S is called a secure dominating set of G. The secure domination number $\gamma_s(G)$ is the minimum cardinaly of a secure dominating set of G. This concept was introduced by Cockayne et al. [7]. It has been further investigated by several authors [1, 2, 3, 5, 6, 9, 10].

Cockayne et al. [7] obtained a characterization of minimal secure dominating sets.

Notation 1.2. *If X is a dominating set of G, let $S=\{v \in X : X - \{v\}$ is a dominating set of $G\}$. For $u \in V-X$, let $A(u,X)=\{v \in X : v$ X-defends $u\}$.*

Theorem 1.3 [7] *A secure dominating set X is minimal if and only if for each $s \in S$ with $N(s) \cap S \neq \phi$ there exists $u_s \in V-X$ such that for each $v \in A(u_s,X)-\{s\}$, one of the following holds:*

1. *There exists $w \in V-X$ such that $N(w) \cap X=\{v,s\}$ and $u_s \notin N(w)$.*

2. *$N(s)\cap X=\{v\}$ and $u_s\in N(v)-N(s)$.*

In this paper we introduce the concept of secure irredundance leading to an inequality chain of four parameters. We present several basic results on these parameters.

2 Main Results

The following definition naturally arises as in the case of domination.

Definition 2.1. *The maximum cardinality of a minimal secure dominating set of G is called the upper secure domination number of G and is denoted by $\Gamma_s(G)$.*

It trivially follows from the definition that $\gamma_s(G)\leq \Gamma_s(G)$. We observe that $\gamma_s(P_5)=\Gamma_s(P_5)=3$ and $\gamma_s(P_7)=3$ and $\Gamma_s(P_7)=4$.

We now proceed to introduce the concept of secure irredundance. For this purpose we need the following notation.

Definition 2.2 *Let $G=(V,E)$ be a graph and let $X\subseteq V$. Let $R=\{v\in X{:}v$ has no X-private neighbor$\}$. Let $v\in X$ and $u\in V-X$. We say that v X-safeguards u, if u is adjacent to all the X-private neighbors of v.*

Notation 2.3. *Let $B(u, X)=\{v\in X{:}v$ X-safeguards $u\}$.*

Proposition 2.4. *Let X be a dominating set of G and let $R=\{v\in X{:}v$ has no X-private neighbor$\}$. Then $R=S$ where S is as defined in Notation 1.2. Further v X-safeguards u if and only if v X-defends u.*

Proof. Let $v\in X$. If $v\in R$, then v has no X-private neighbor and hence $N(v)\cap X\neq\phi$. Hence $X-\{v\}$ is a dominating set of G, so that $v\in S$. Thus $R\subseteq S$. By a similar argument $S\subseteq R$ and so $R=S$. Now if v X-safeguards u, then u is adjacent to all the X-private neighbors of v. Hence $(X-\{v\})\cup\{u\}$ is a dominating set of G, so that v X-defends u. The proof of the converse is similar. □

Observation 2.5. *It follows from Proposition 2.4 that $B(u,X)=A(u,X)$ where $A(u,X)$ is as defined in Notation 1.2.*

Definition 2.6. *Let $G=(V,E)$ be a graph and let $X\subseteq V$. Then X is called a secure irredundant set if for every $r\in R$ with $N(r)\cap R\neq\phi$, there exists $u_r\in V-X$ such that for each $v\in B(u_r,X)-\{r\}$, one of the following holds.*

1. *There exists $w\in V-X$ such that $N(w)\cap X=\{v,r\}$ and $u_r\notin N(w)$.*
2. *$N(r)\cap X=\{v\}$ and $u_r\in N(v)-N(r)$.*

Theorem 2.7. *Secure irredundance is a hereditary property.*

Proof. Let X be a secure irredundant set of G. Let $Y\subseteq X$ and let $u\in V-Y$. Let $R_1=\{v\in Y{:}v$ has no Y-private neighbor$\}$ and $B_1(u,Y)=\{v\in Y{:}v$ Y-safeguards $u\}$. Now let $v\in R_1$. Then v has no Y-private neighbor and hence has no X-private neighbor.

$$\text{Thus } R_1 \subseteq R \qquad (1)$$

Now $PN(v,X) \subseteq PN(v,Y)$ for all $v \in Y$. Hence if a vertex u of $V-X$ is adjacent to all vertices in $PN(v,Y)$, then it is adjacent to all vertices of $PN(v,X)$.

$$\text{Thus } B_1(u,Y) \subseteq B(u,X) \text{ for all } u \in V-X. \qquad (2)$$

Now let $r \in R_1$ and $N(r) \cap R_1 \neq \phi$. It follows from (1) that $r \in R$ and $N(r) \cap R \neq \phi$. Now since X is secure irredundant, there exists $u_r \in V-X$ such that for all $v \in B(u,X)-\{r\}$, one of the following holds.

(i) There exists $w \in V \setminus R$ such that $N(w) \cap X = \{v,r\}$ and $u_r \notin N(w)$.

(ii) $N(r) \cap X = \{v\}$ and $u_r \in N(v)-N(r)$.

Clearly, $u_r \in V-Y$. Now let $v \in B_1(u,Y)-\{r\}$. It follows from (2) that $v \in B(u,X)-\{r\}$ and since X is secure irredundant, (i) or (ii) holds for v. Hence Y is secure irredundant. □

It follows from Theorem 2.7 that a secure irredundant set X is maximal if and only if $X \cup \{v\}$ is not secure irredundant for all $v \in V-X$.

Definition 2.8. *The secure irredundance number $ir_s(G)$ and the upper secure irredundance number $IR_s(G)$ are defined by*

$ir_s(G) = min\{|X| : X$ is a maximal secure irredundance set of $G\}$ and

$IR_s(G) = max\{|X| : X$ is a maximal secure irredundance set of $G\}$.

Theorem 2.9. *A secure dominating set X is a minimal secure dominating set if and only if X is secure dominating and secure irredundant.*

Proof. The result follows from Theorem 1.3 and Proposition 2.4. □

Corollary 2.10. *For any graph G we have* $ir_s(G) \leq \gamma_s(G) \leq \Gamma_s(G) \leq IR_s(G)$.

3 Conclusion and Scope

In this paper we have introduced an inequality chain of four parameters arising from secure domination and secure irredundance. However, the standard domination chain is an inequality chain of six parameters arising from independence, domination and irredundance. Hence the following problem arises naturally.

Problem 3.1. *Define a graph theoretic property P such that P is hereditary and a subset S of V is a maximal P-set if and only if S is a P-set and S is a minimal secure dominating set.*

Such a property P which is analogous to independence in the context of domination can be used to construct the secure domination chain.

The following is another interesting problem for further investigation.

Problem 3.2. *Given four positive integers a,b,c and d with* $a \leq b \leq c \leq d$, *under what conditions there exists a graph G with* $ir_s(G)=a$, $\gamma_s(G)=b$, $\Gamma_s(G)=c$ *and* $IR_s(G)=d$?

ACKNOWLEDGEMENTS
This research work is supported by RUI grant # 1001/PKOMP/811290 awarded by Universiti Sains Malaysia.

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