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Crossing numbers of join of a graph on six vertices with a path and a cycle

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Abstract

The crossing number of a graph G is the minimum number of crossings of its edges among the drawings of G in the plane and is denoted by cr(G). Zarankiewicz conjectured that the crossing number of the complete bipartite graph $K_{m,n}$ equals $\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$. This conjecture has been verified by Kleitman for min {m, n} $\leq$ 6. Using this result, we give the exact values of crossing number of the join of a certain graph G on six vertices with a path and a cycle on n vertices.

Keywords: *Crossing Number, Good drawing of a graph, Union and join of graphs*

1 Introduction

Let $G(V, E)$ be a simple connected undirected graph with vertex set V and edge set E. A drawing D of a graph G is a representation of G in the Euclidean plane where vertices are represented by distinct points and edges by simple polygonal arcs joining the points that correspond to their end vertices. A drawing D is *good* or *clean* if no edge crosses itself, no pair of adjacent edges cross, two edges cross at most once and no more than two edges cross at one point.

The crossing number of a graph G is the smallest number of edge crossings in any drawing of G and it is denoted by $cr(G)$. It is well known that the crossing number of a graph is attained only in good drawings of the graph. So, we always assume that all drawings throughout this paper are good. Further it is clear that G is planar if and only if $cr(G) = 0$.

Crossing number minimization is one of the fundamental optimization problems in the sense that it is related to various other widely used notions. There are numerous applications, most notably those in VLSI design and in combinatorial geometry [1, 16, 19]. Researchers in computer science focus their attention to this area of graph theory as the study of crossing numbers of graphs finds applications in network design and circuit layout. It is also an important measure of non-planarity of a graph.

Crossing number problems were introduced by Turán [18], who first inquired about the crossing number of the complete bipartite graph $K_{m,n}$. Zarankiewicz devised a natural drawing of $K_{m,n}$ with $\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ crossings, but his conjecture [20], that such a drawing is the best possible, is still open [14]. Garey and Johnson [4] proved that computing the crossing number is *NP*-complete. The crossing number problem for generalized Petersen graphs has been investigated in [3] and [17]. Determining the exact values of the crossing number for graphs is a challenging problem. Join and Cartesian products of graphs are two graph families for which exact results concerning the crossing numbers are known [14].

Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be any two graphs. Their *union*, denoted by $G_1 \cup G_2$ is the graph $(V_1 \cup V_2, E_1 \cup E_2)$. For any two vertex disjoint graphs G_1 and G_2, their *join*, denoted by $G_1 + G_2$, is obtained from $G_1 \cup G_2$ by joining every vertex of G_1 to every vertex of G_2. When $|V(G_1)| = m$ and $|V(G_2)| = n$, the edge set of $G_1 + G_2$ is the union of disjoint edge sets of G_1 and G_2 and that of the complete bipartite graph $K_{m,n}$. The conjecture of Zarankiewicz [20] has been verified by Kleitman [5] for min $\{m, n\} \leq 6$. Klešč [10] has given the exact values of the crossing number for join of two paths, join of two cycles and for join of a path and a cycle. In addition, he has given the exact values for crossing number of $G + P_n$ and $G + C_n$ for all graphs G of order at most four [6]. He has also obtained this result for some of the graphs on five and six vertices [7 – 15].

Let D be a good drawing of a graph G. We denote the number of crossings in D by $cr_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote the number of crossings between the edges of G_i and the edges of G_j by $cr_D(G_i, G_j)$.

Lemma 1 [5] *Let G_i, G_j and G_k be mutually edge-disjoint subgraphs of G. Then*

$$cr_D(G_i \cup G_j) = cr_D(G_i) + cr_D(G_j) + cr_D(G_i, G_j) \quad (1)$$

$$cr_D(G_i \cup G_j, G_k) = cr_D(G_i, G_k) + cr_D(G_j, G_k) \quad (2)$$

The following result on the crossing number of a complete bipartite graph $K_{m,n}$ is due to Kleitman.

Lemma 2 [5] *For min* $\{m, n\} \leq 6$, $cr(K_{m,n}) = Z(m, n)$ *where*

$$Z(m,n) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Theorem 1 [2] (*Euler's Formula*) *If G is connected plane graph, then* $v - \in + \phi = 2$, *where v is the number of vertices,* $\in$ *is the number of edges and* ϕ *is the number of faces in G.*

Corollary 2 [2] *If G is a connected plane graph with girth g, the* $g\phi \leq 2\in$.

Let G be the graph with vertex set $\{a, b, c, d, e, f\}$ and edge set $\{ab, ac, ae, af, bc, bf, cd, ce, de, ef\}$ as shown in Fig.1(*i*). It consists of a 6-cycle $C_6(G) = abcdefa$, five 3-cycles, *abca*, *afea*, *abfa*, *ced*c, *acea*, and one 4-cycle, *bcefb*. The graph in Fig.1(*ii*) is a good drawing of *G*.

In this paper, we give the exact values of crossing number of the join of the graph *G* with a path and a cycle on *n* vertices. We follow the notations and proof techniques given by Klešč et al. [13, 14] in our paper. As mentioned there the term ″region″ in a nonplanar drawing is to be understood in the sense crossings are vertices of the ″map″. We consider first the join of *G* with the discrete graph on *n* vertices.

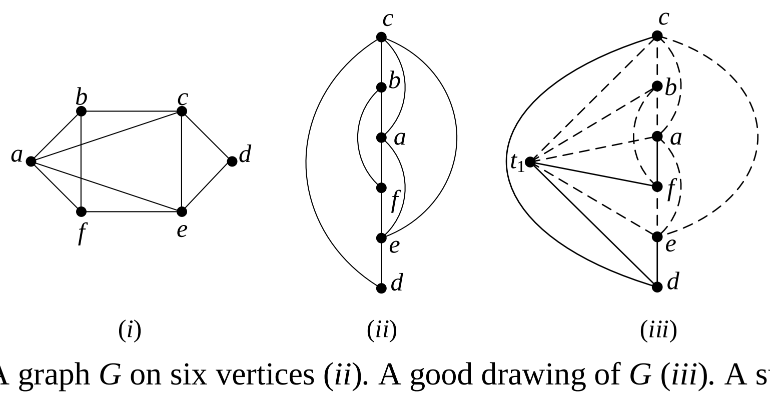

Fig. 1: (*i*). A graph *G* on six vertices (*ii*). A good drawing of *G* (*iii*). A subdivision of K_5 in $G + K_1$

2 The Crossing Number of $G + nK_1$

Zarankiewicz [20] gave a drawing of $K_{m,n}$ which meant that $cr(K_{m,n}) \leq Z(m, n)$. The graph $G_n = G + nK_1$ consists of one copy of the graph G and n vertices t_1, t_2 … t_n of nK_1, where each vertex t_i, $i = 1, 2 \ldots n$, is adjacent to every vertex of G. Place $\left\lceil \frac{n}{2} \right\rceil$ of these vertices to the negative position on the x-axis, $\left\lfloor \frac{n}{2} \right\rfloor$ of them to the positive position on the x-axis, 3 vertices of G to the negative position on y-axis, 3 of them to the positive position on the y-axis and draw $6n$ edges by straight line segments to obtain a drawing of $K_{6,n}$. Let T^i, $1 \leq i \leq n$, denote the subgraph induced by the six edges incident with the vertex t_i. Then

$$G_n = G + nK_1 = G \cup K_{6,n} = G \cup (\cup_{i=1}^{n} T^i).$$

Lemma 3 $cr(G + K_1) = 1$.

Proof. It is clear from Fig.1(iii) that $cr_D(G + K_1) \leq 1$. Further $G + K_1$ contains a subdivision of K_5 and hence $cr_D(G + K_1) \geq 1$.

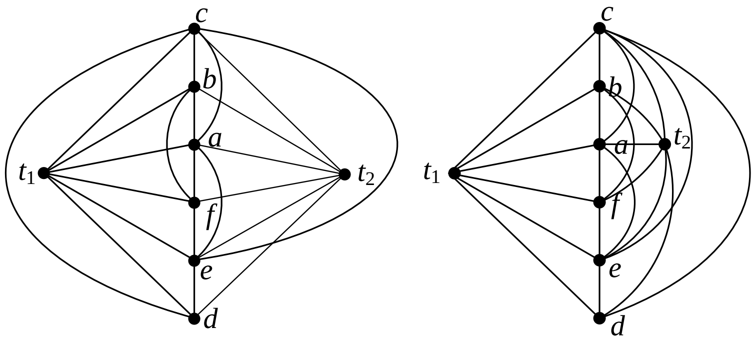

Fig. 2: Possible diagrams of G+2K_1

Lemma 4 $cr(G + 2K_1) = 4$.

Proof. In the graph $G + 2K_1$, let r be the smallest nonnegative integer such that the removal of some r edges from the graph $G + 2K_1$ results in a planar subgraph $(G + 2K_1)_r$ of $(G + 2K_1)$. This graph is a connected spanning subgraph of $G + 2K_1$ with eight vertices and $22 - r$ edges. By Theorem 1, $\phi = 16 - r$. Since $(G + 2K_1)_r$ has girth at least three, we have by Corollary 2, $3(16 - r) \leq 2(22 - r)$, or $4 \leq r$. Thus $cr_D(G + 2K_1) \geq 4$. It follows from the drawing in Fig. 2 that $cr_D(G + 2K_1) \leq 4$ and therefore $cr(G + 2K_1) = 4$.

Lemma 5 *Let D be a good drawing of $G + nK_1$, $n > 1$, in which there are two different subgraphs T^i and T^j with $cr_D(T^i, T^j) = 0$. Then there are at least $Z(6, n)$ + $n + 2\left\lfloor \frac{n}{2} \right\rfloor$ crossings in D.*

Proof. We use induction on n for the proof. By Lemma 4, the result is true for $n = 2$. Let $n \geq 3$ and without loss of generality let $cr_D(T^{n-1}, T^n) = 0$. Assume further that for every integer $s < n$, any good drawing of $G + sK_1$ has at least $Z(6, s) + s + 2\left\lfloor\frac{s}{2}\right\rfloor$ crossings. If possible let D be a good drawing $G_n = G + nK_1$ with less than $Z(6, n) + n + 2\left\lfloor\frac{n}{2}\right\rfloor$ crossings. The subdrawing of D induced by $T^{n-1} \cup T^n$ with $cr_D(T^{n-1}, T^n) = 0$ divides the plane into several regions such that the boundary of each region has exactly two vertices of G. Hence, in D, the edges of G cross the edges of $T^{n-1} \cup T^n$ at least four times; that is $cr_D(G, T^{n-1} \cup T^n) \geq 4$. Further, since $cr(K_{6,3}) = 6$, we have $cr_D(T^k, T^{n-1} \cup T^n) \geq 6$, for $k = 1, 2 \ldots n-2$. Again, since $G_n = G_{n-2} \cup (T^{n-1} \cup T^n)$, we have

$$cr_D(G_n) = cr_D(G_{n-2} \cup (T^{n-1} \cup T^n))$$

$$= cr_D(G_{n-2}) + cr_D(T^{n-1} \cup T^n) + cr_D(G_{n-2}, T^{n-1} \cup T^n), \text{ by Lemma 1}$$

$$\geq Z(6, n-2) + (n-2) + 2\left\lfloor\frac{n-2}{2}\right\rfloor + 0 + cr_D(G_{n-2}, T^{n-1} \cup T^n), \text{ by hypothesis}$$

$$= Z(6, n-2) + (n-2) + 2\left\lfloor\frac{n-2}{2}\right\rfloor + cr_D\left(G \cup \left(\bigcup_{i=1}^{n-2} T^i\right), T^{n-1} \cup T^n\right)$$

$$= Z(6, n-2) + (n-2) + 2\left\lfloor\frac{n-2}{2}\right\rfloor + cr_D(G, T^{n-1} \cup T^n)$$

$$+ cr_D\left(\left(\bigcup_{i=1}^{n-2} T^i\right), T^{n-1} \cup T^n\right), \text{ by Lemma 1}$$

$$\geq Z(6, n-2) + (n-2) + 2\left\lfloor\frac{n-2}{2}\right\rfloor + 4 + 6(n-2)$$

$$= 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + n + 2\left\lfloor\frac{n}{2}\right\rfloor$$

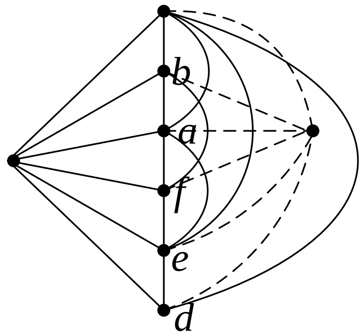

Fig. 3: The crossing of the graph $G \cup T^i$ with T^j

contrary to assumption that G has less than $Z(6, n) + n + 2\left\lfloor \frac{n}{2} \right\rfloor$ crossings.

Lemma 6 *Let D be a good drawing of $G + nK_1$, $n \geq 3$, in which some subgraph $G \cup T^i$ is crossed at least seven times by every subgraph T^j, $j = 1, 2, \ldots n$, $j \neq i$. Then there are at least $Z(6, n) + n + 2\left\lfloor \frac{n}{2} \right\rfloor$ crossings in D.*

Proof. Clearly $G + nK_1 = K_{6,n-1} \cup (G \cup T^i)$ and $K_{6,n-1}$ contains $n - 1$ subgraphs T^j, $j \neq i$. Consequently

$$cr_D(G + nK_1) = cr_D(K_{6,n-1}) + cr_D(G \cup T^i) + cr_D(K_{6,n-1}, G \cup T^i)$$

$$\geq Z(6, n-1) + 1 + 7(n-1), \text{ by assumption, see Fig. 3}$$

$$= 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 7n - 6$$

$$= Z(6, n) + n + 6\left\lfloor \frac{n}{2} \right\rfloor$$

$$\geq Z(6, n) + n + 2\left\lfloor \frac{n}{2} \right\rfloor$$

We are now ready to prove the main theorem of this section.

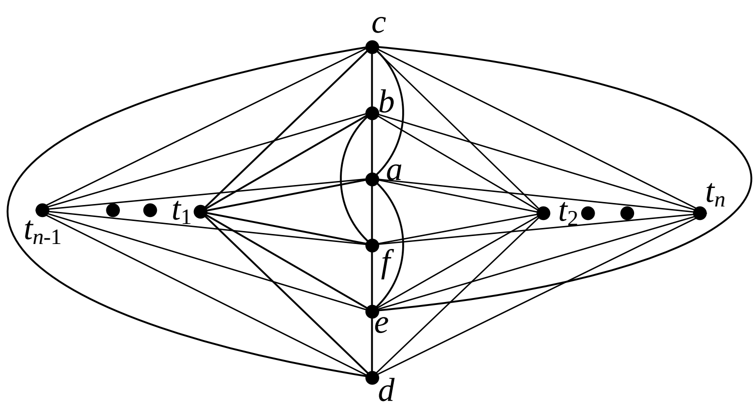

Fig. 4: A good drawing of $G + nK_1$

Theorem 3 *$cr(G + nK_1) = Z(6, n) + n + 2\left\lfloor \frac{n}{2} \right\rfloor$ for $n \geq 1$.*

Proof. The theorem is true for $n = 1$ and $n = 2$. For $n \geq 3$, let D be good drawing of $G_n = G + n\,K_1$ as in Fig. 4. Apart from the crossings of $K_{6,n}$, the edges ce, ca, ae and bf contribute for the crossing number of D. The first three edges contribute $\left\lfloor \frac{n}{2} \right\rfloor$ crossings each and the edge bf contributes $\left\lceil \frac{n}{2} \right\rceil$ crossings to the crossing number of

D. Hence it follows from Fig. 4 that $cr_D(G_n) \leq Z(6, n) + n + 2\left\lfloor\frac{n}{2}\right\rfloor$. Consider a good drawing D of G_n with less than $Z(6, n) + n + 2\left\lfloor\frac{n}{2}\right\rfloor$ crossings. Since

$$\begin{aligned} cr_D(G_n) &= cr(G \cup K_{6,n}) \\ &= cr_D(G) + cr_D(K_{6,n}) + cr_D(G, K_{6,n}) \\ &\geq cr_D(G) + \mathrm{Z}(6,n) + cr_D(G, K_{6,n}) \end{aligned}$$

our assumption on D implies that

$$cr_D(G) + cr_D(G, K_{6,n}) < n + 2\left\lfloor\frac{n}{2}\right\rfloor. \tag{3}$$

Hence the edges of G are crossed less than $n + 2\left\lfloor\frac{n}{2}\right\rfloor$ times in D. This implies that, in D, there is at least one subgraph T^i which does not cross G. Without loss of generality let $cr_D(G, T^n) = 0$.

The choice of D along with Lemma 5 implies that $cr_D(T^i, T^j) \neq 0$ for all $i, j = 1, 2 \ldots n$, $i \neq j$. Further it follows from Lemma 6 that $cr_D(G \cup T^n, T^i) \leq 6$ for at least one subgraph T^i, $i \in \{1, 2 \ldots n-1\}$.

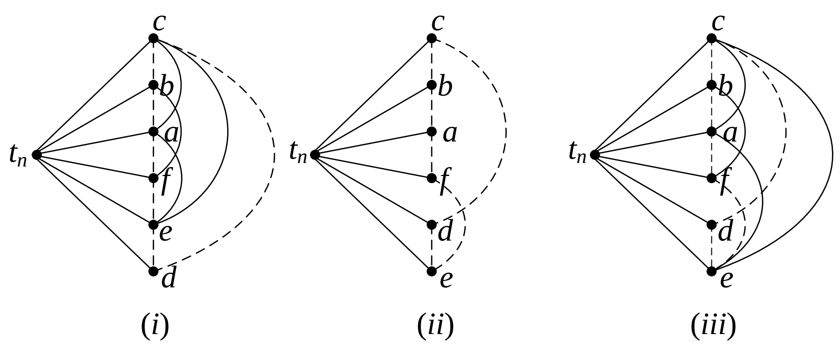

Fig. 5: (i) A subdrawing of $G \cup T^n$ in which the edges of the 6-cycle do not cross each other. (ii). A subdrawing of $C_6(G) \cup T^n$ in which the edges of the 6-cycle cross each other. (iii). A subdrawing of $G \cup T^n$ in which the edges of the 6-cycle cross each other.

Consider the subdrawing D* *of* G $\cup$ T^n *induced by* D. *We consider two cases.*

Case 1: The edges of the 6-cycle $C_6(G)$ do not cross each other in D^*.

The cycle $C_6(G)$ determines two regions in D^*, say the interior and the exterior of $C_6(G)$. Since $cr_D(G, T^n) = 0$, we assume without loss of generality that all the edges of T^n lie in, say, the exterior of $C_6(G)$. The remaining edges of G not belonging to $C_6(G)$ are placed in the interior of $C_6(G)$. One such a drawing is shown in Fig. 5.

It is easy to see that if some vertex t_i, $i = 1, 2 \ldots n - 1$, is placed inside any of the 3-cycles *abca*, *afea*, *cedc*, and *abfa*, then the edges of T^i cross the edges of G at least six times. Since $cr_D(T^i, T^j) \neq 0$ for all $i, j = 1, 2 \ldots n$, $i \neq j$, this means that $cr_D(G \cup T^n, T^i) \geq 7$. If t_i is placed inside the 3-cycle *acea* or the 4-cycle *bcefb*, then the edges of T^i cross the edges of G at least four times; consequently $cr_D(G \cup T^n, T^i) \geq 5$. Moreover if t_i is placed in a region outside G, then $cr_D(G \cup T^n, T^i) \geq 10$ and if none of the edges of T^i crosses G, then $cr_D(G \cup T^n, T^i) \geq 6$.

Let r be the number of vertices t_i, $i = 1, 2 \ldots n - 1$, which are placed, in D, in the region bounded by the triangles *abca*, *afea*, *cedc*, and *abfa* for which $cr_D(G \cup T^n, T^i) \geq 7$. Since there is a subgraph T^i with $cr_D(G \cup T^n, T^i) \leq 6$, we have $r \geq 1$. Let s be number of vertices t_i, $i = 1, 2 \ldots n - 1$, which are placed, in D, in the region bounded by 3-cycle *acea* or the 4-cycle *bcefb*, for which $cr_D(G \cup T^n, T^i) \geq 5$. Thus

$$7r + 5s < n + 2\left\lfloor \frac{n}{2} \right\rfloor.$$

$$\begin{aligned} cr_D(G + nK_1) &= cr_D(K_{6,n-1}) + cr_D(G \cup T^n) + cr_D(K_{6,n-1}, G \cup T^n) \\ &\geq Z(6, n-1) + 7r + 5s + 6(n - r - s - 1) \\ &= Z(6, n) + r - s + 6\left\lfloor \frac{n}{2} \right\rfloor \end{aligned}$$

Since D is a good drawing of G_n with less than $Z(6, n) + n + 2\left\lfloor \frac{n}{2} \right\rfloor$ crossings, we have $Z(6, n) + n + 2\left\lfloor \frac{n}{2} \right\rfloor > Z(6, n) + r - s + 6\left\lfloor \frac{n}{2} \right\rfloor$. This gives $r - s < n - 4\left\lfloor \frac{n}{2} \right\rfloor$ which together with the condition $7r + 5s < n + 2\left\lfloor \frac{n}{2} \right\rfloor$ results in $r < \frac{1}{2}\left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor$. A contradiction to the fact that $r > 1$.

Case 2: The edges of the 6-cycle $C_6(G)$ cross each other in D^*.

We know that there is a subgraph T^i such that and $cr_D(G \cup T^n, T^i) \leq 6$ and that $cr_D(T^n, T^i) \neq 0$ for i, $1 \leq i \leq n - 1$. Thus $cr_D(G \cup T^n, T^i) \leq 6$ means that $cr_D(C_6(G), T^i) \leq 5$. Since $cr_D(C_6(G), T^n) = 0$, the vertex t_n is placed in a region with all six vertices of $C_6(G)$ on its boundary and the conditions $cr_D(C_6(G), T^i) \leq 5$ means that in the subdrawing of $C_6(G) \cup T^n$ there is a region with at least 2 vertices on its boundary. When the edges of G are included this gives $cr_D(G \cup T^n, T^i) \geq 6$, a contradiction.

So there is no good drawing of G_n with less than $Z(6, n) + n + 2\left\lfloor \frac{n}{2} \right\rfloor$ crossings.

3 Crossing number of $G + P_n$ and $G + C_n$

The graph $G + P_n$ contains $G + nK_1$ as a subgraph. For the subgraphs of $G + P_n$ which are also subgraphs of $G + nK_1$ we use the same notation as above. Let P_n^* denote the path on n vertices of $G + P_n$ not belonging to the subgraph G. One can easily see that $G + P_n = G \cup K_{6,n} \cup P_n^*$

It is clear that, for $n = 1$, the graph $G + P_1$ is isomorphic to $G + K_1$ and hence $cr_D(G + P_1) = 1$.

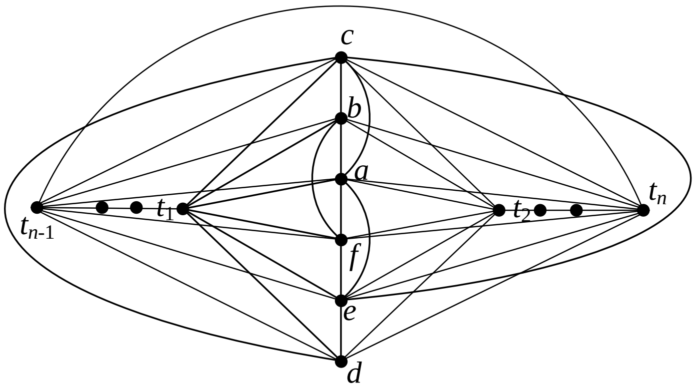

Fig. 6: The graph $G + P_n$

Theorem 4 $cr(G + P_n) = Z(6, n) + n + 2\left\lfloor \frac{n}{2} \right\rfloor + 2$*, for* $n \geq 1$.

Proof. It is clear from Fig. 6 that $cr_D(G + P_n) \leq Z(6, n) + n + 2\left\lfloor \frac{n}{2} \right\rfloor + 2$. Suppose $cr_D(G + P_n) < Z(6, n) + n + 2\left\lfloor \frac{n}{2} \right\rfloor + 2$. By removing the edge $t_{n-1}t_n$ which contributes 2 crossings, the resulting graph $(G + P_n)_r$ will have fewer than $Z(6, n) + n + 2\left\lfloor \frac{n}{2} \right\rfloor$ crossings. Also, $G + nK_1 \subset (G + P_n)_r$ and

$$\begin{aligned} cr_D(G + nK_1) &< cr_D(G + P_n)_r \\ &< Z(6, n) + n + 2\left\lfloor \frac{n}{2} \right\rfloor \end{aligned}$$

This contradicts Theorem 3.

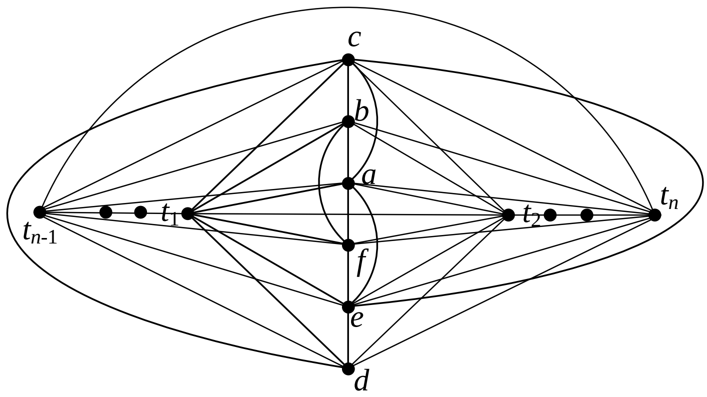

Fig. 7: The graph $G + C_n$

Similarly we have the following result whose proof is omitted.

Theorem 5 $cr(G + C_n) = Z(6, n) + n + 2\left\lfloor \frac{n}{2} \right\rfloor + 5$

4 Conclusion

In this paper we have obtained the exact values of crossing number of $G + nK_1$, $G + P_n$ and $G + C_n$, where G is a graph on six vertices as shown in Fig.1(*i*). There are other graphs on 6 vertices for which the problem remains open.

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References

[1] Bhatt, S. N., Leighton, F. T., 1984. A framework for solving VLSI graph layout problem, *J. Comput. System Sci.*, Vol.28, 300-343.

[2] Bondy, J. A., Moorthy, U. S. R., 1982. Graph Theory with Applications, MacMillan Press.

[3] Exoo, G., Harary, F., Kabell, J., 1981. The crossing number of some generalized Petersen graph, *Math. Scand.*, Vol.48, 184-188.

[4] Garey, M. R., Johnson, D. S., 1983. Crossing Number is NP-Complete, *SIAM J. Alg. Disc. Meth.*, Vol.4, 312-316.

[5] Kleitman, D. J., 1971. The crossing number of $K_{5,n}$, *J. Combin. Theory,* Vol.9, 315-323.

[6] Klešč, M., 1994. The crossing numbers of products of paths and stars with 4-vertex graphs, *J. Graph Theory,* Vol.18, 605-614.

[7] Klešč, M., 2001. On the crossing numbers of products of stars and graphs of order five, *Graphs and Comb.,* Vol.17, 289-294.

[8] Klešč, M., 2001. The crossing numbers of Cartesian products of paths with 5-vertex graphs, *Discrete Math.,* Vol.233, 353-359.

[9] Klešč, M., 2005. Some crossing numbers of products of cycles, *Discuss. Math. Graph Theory,* Vol.25, 197-210.

[10] Klešč, M., 2007. The join of graphs and crossing numbers, *Electronic Notes in Discrete Math.*, Vol.28, 349-355.

[11] Klešč, M., Kocúrová, A., 2007. The crossing number of products of 5-vertex graphs with cycles, *Discrete Math.,* Vol.307, 1395-1403.

[12] Klešč, M., 2009. On the crossing numbers of Cartesian products of stars and graphs of order five, Combinatorial Algorithms, Springer, LNCS 5874, 324-333.

[13] Klešč, M., 2010. The crossing number of the join of special graph on six vertices with path and cycle, *Discrete Mathematics*, Vol.310, 1475-1481.

[14] Klešč, M., Kraveková, D., Petrilová, J., 2011. The crossing numbers of the join of special graphs, *Proc. Faculty of Electrical Engineering Informatics, Technical University of Košice,* 1-6.

[15] Klešč, M., Stefan, S., 2012. The crossing numbers of join of paths and cycles with two graphs of order five, *Lecture Notes in Computer Science*, Vol.7125, 160-167.

[16] Leighton, F. T., 1984. New lower bound techniques for VLSI, *Math. Systems Theory,* Vol.17, 47-70.

[17] Richter, B. R., Salazar, G., 2002. The crossing number of *P*(*n*, 3), *Graphs and Combinatorics,* Vol.18, 381-394.

[18] Turán, P., 1977. A note of welcome, *J. Graph Theory,* Vol.1, 7-9.

[19] Székely, L. A., 2004. A successful concept for measuring nonplanarity of graphs: the crossing number, *Discrete Mathematics,* Vol.276, 331-352.

[20] Zarankiewicz. K., 1954. On a problem of P. Turán concerning graphs, *Fund. Math,* Vol.41, 137-145.