Int. J. Advance Soft Compu. Appl, Vol. 8, No. 2, July 2016
ISSN 2074-8523

Eccentric Digraphs of Tournaments

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Abstract

Let $G=(V,A)$ be a digraph. The eccentricity $e(u)$ of a vertex u is the maximum distance from u to any other vertex in G. A vertex v in G is an eccentric vertex of u if the distance from u to v equals $e(u)$. The eccentric digraph $ED(G)$ of a digraph G has the same vertex set as G and has arcs from a vertex v to its eccentric vertices. In this paper we present several results on the eccentric digraph of a tournament.

Keywords: *Eccentricity, Eccentric Vertex, Eccentric Digraph, Tournament*

1 Introduction

Buckley [3] introduced the notion of *eccentric digraph of a graph* which was then refined by others, including Boland and Miller [1]. In [8] the iteration of distance digraph of a graph is discussed. In this paper, we consider the *eccentric digraph* of tournaments and obtain several properties. We also derive conditions that ensure that the dual of a tournament T is the eccentric digraph of T.

By a graph $G=(V,E)$, we mean a finite, undirected graph with neither loops nor multiple edges. Similarly in a digraph $G = (V,A)$, multiple arcs or loops are not allowed. For graph theoretic terminology in graphs and digraphs we refer to Chartrand and Lesniak [5]. The order $|V|$ and the size $|E|$ of G are denoted by n and m respectively.

Let $G=(V,A)$ be a digraph and let $u\in V$. Then $O(u)=\{v\in V: (u,v)\in A\}$ and $I(u)=\{v\in V: (v,u)\in A\}$ are respectively called out-neighbor set and in-neighbor set of u. Also $|O(u)|=d^+(u)$ is called the out-degree of u and $|I(u)|=d^-(u)$ is called the in-degree of u.

The *eccentric digraph* $ED(G)$ of a digraph G is the digraph on the same vertex set as that of G with an arc from vertex u to vertex v in $ED(G)$ if and only if v is an eccentric vertex of u in G. For every digraph G there exist smallest positive integers $p>0$ and $t\geq 0$ such that $ED^t(G) = ED^{p+t}(G)$. The integers p and t are called the period of G and the tail of G and are denoted by $p(G)$ and $t(G)$ respectively. If $t=0$ then G is called periodic.

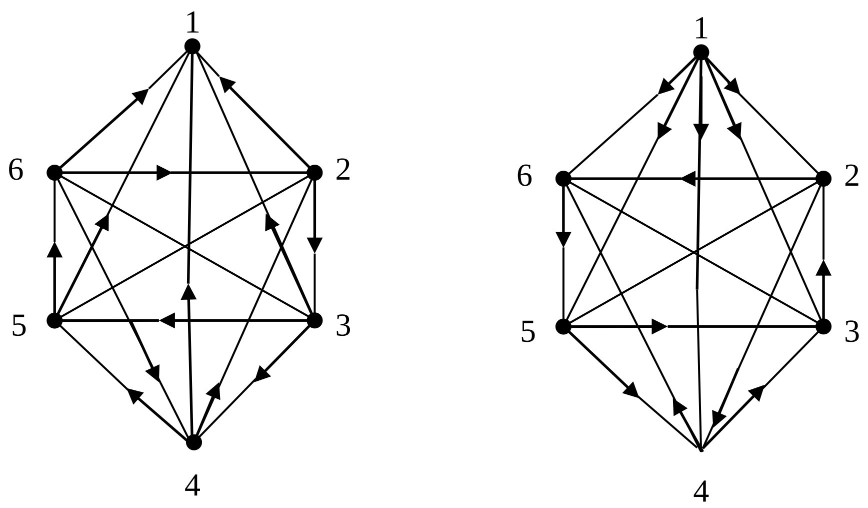

Fig. 1: Tournament T and its Eccentric Digraph $ED(T)$

For several basic results on eccentric digraphs we refer to [2,4,6,7].

A graph G is self-centered if $e(v)=rad(G) = diam(G)$ for all vertices v of G.

A tournament T is a complete asymmetric digraph. In other words, for every two distinct vertices $u,v \in V(T)$, either $(u, v) \in A(T)$ or $(v,u)\in A(T)$, but not both, and (v, v) is not in $A(T)$ for all $v\in V(T)$. A tournament with n vertices, is called an n-tournament. The score of a vertex u of a tournament is its out-degree and it is denoted by $s(u)$ or $s_T(u)$. The score sequence of a tournament is the nondecreasing sequence of out-degrees of the vertices of the tournament. The score set of a tournament is the set of integers that are the out-degrees of vertices in that tournament. A regular tournament is one in which every vertex has the same out-degree. A tournament is transitive if whenever (u,v) and (v,w) are arcs in T, then (u,w) is also an arc in T. The dual of T is the tournament T^r, with $V(T^r) = V(T)$ and $(u,v) \in A(T^r)$ if and only if $(v,u) \in A(T)$. We define the dual of a digraph G in the same way and use the same notation.

We need the following theorem.

Theorem 1.1. ([5], Page 139) *A nondecreasing sequence S of $n(\geq 1)$ nonnegative integers is a score sequence of a transitive tournament of order n if and only if S is the sequence $0,1,\ldots,n-1$.*

2 Eccentric Digraph of a Tournament

In this section we obtain several results on eccentric digraphs of tournaments.

Lemma 2.1. *Let $T=(V,A)$ be a tournament on n vertices and let $u \in V(T)$. Then $s_T(u) = n-1$ if and only if $\deg^+_{ED(T)}(u) = n-1 = \deg^-_{ED(T)}(u)$.*

Proof. Let $u \in V(T)$. Then $s_T(u) = n-1$ if and only if all the vertices in $T-\{u\}$ are eccentric vertices of u and u is the eccentric vertex of all the vertices in $T-\{u\}$. Hence $s_T(u) = n-1$ if and only if $\deg^+_{ED(T)}(u) = n-1 = \deg^-_{ED(T)}(u)$. □

Corollary 2.2. *If T is a tournament having a source w, then $ED(T)$ is a strongly connected digraph but is not a tournament.*

Theorem 2.3 *Let $T=(V,A)$ be a tournament on n vertices. Then $ED(T) = T^r$ if and only if for all u in $V(T)$, the following conditions are satisfied.*

(i) $s_T(u) \neq n-1$.

(ii) $d_T(u,v) = 2$ for all $v \in N^-_T(u)$ or $d_T(u,v) = \infty$, for all $v \in N^-_T(u)$.

Proof. Let $T=(V,A)$ be a tournament on n vertices satisfying the conditions (i) and (ii). Let $u \in V(T)$. Then condition (i) implies that $N^-_T(u) \neq \phi$ and condition (ii) implies that every in-neighbor v of u in T is an eccentric vertex of u. Hence v is an out-neighbor of u in $ED(T)$. Also, if w is an out-neighbour of u in T, then u is an in-neighbour of w in T. Hence u becomes an out-neighbour of w in $ED(T)$. Thus $ED(T) = T^r$.

Conversely, suppose $ED(T) = T^r$. Let $u \in V(T)$. If $s_T(u) = n-1$, then by Lemma 2.1 we have $\deg^+_{ED(T)}(u) = n-1 = \deg^-_{ED(T)}(u)$, which contradicts the assumption that $ED(T) = T^r$. Therefore $s_T(u) \neq n-1$ for all u in $V(T)$. This proves (i). Now, let $s_T(u) = k$ and $N^-_T(u) = \{v_1, v_2, v_3, \ldots, v_{n-k-1}\}$. Since $ED(T) = T^r$, each v_i is an eccentric vertex of u. Hence $d_T(u,v_i) = d_T(u,v_j)$ for all $v_i, v_j \in N_T^-(u)$. Now, $d_T(u,v_i) = 2$ if there exists $w \in N_T^+(u) \cap N_T^-(v_i)$. Otherwise $d_T(u,v_i) = \infty$. This proves (ii). □

Corollary 2.4. *Let $T = (V,A)$ be a tournament. Then $ED(T) = T^r$ and $ED^2(T) = T$ if T is self-centered with radius two.*

Proof. Let T be a self-centered tournament with radius two. Then for every u in $V(T)$, we have $0 < s_T(u) < n-1$ and $d_T(u,v) = 2$ for all $v \in N_T^-(u)$. Hence it follows from Theorem 2.3 that $ED(T) = T^r$. We now claim that T^r is also a self-centered tournament with radius two. Suppose $(u,v) \notin A(T^r)$. Then $(u,v) \in A(T)$ and $(v,u) \notin E(T)$. Hence $d_T(u,v)=2$. Now if (v,w,u) is a path in T, then (u,w,v) is a path in T^r and hence $d_{T^r}(v,u) = 2$. Thus T^r is a self-centred tournament with radius 2. Hence $ED(T^r) = (T^r)^r = T$. Thus $ED^2(T) = T$. □

We now proceed to obtain an upper bound for the number of arcs in the eccentric digraph of a tournament.

Theorem 2.5 *Let $T = (V,A)$ be a tournament on n vertices. If T has no source, then $|A(ED(T))| \le {}^nc_2$ and equality holds if and only if for all $u \in V(T)$ either $d_T(u,v) = 2$ for all $v \in N_T^-(u)$ or $d_T(u,v) = \infty$ for all $v \in N_T^-(u)$.*

Proof. Let $u \in V(T)$. Since T has no source, only in-neighbors of u can be eccentric vertices of u. Hence $|E(u)| \le |N_T^-(u)|$, where $E(u)$ denotes the set of all eccentric vertices of u. This implies that

$$\sum_{u \in V(T)} |E(u)| \le \sum_{u \in V(T)} |N_T^-(u)| = {}^n c_2.$$

Since $|A(ED(T))| = \sum_{u \in V(T)} |E(u)|$, we have $|A(ED(T))| \le {}^nc_2$. If for all $u \in V(T)$, $d_T(u,v) = 2$, for all $v \in N_T^-(u)$ or $d_T(u,v) = \infty$, for all $v \in N_T^-(u)$, then by Theorem 2.3, $ED(T) = T^r$. This implies that $|A(ED(T))| = {}^nc_2$.

Conversely, suppose that $|A(ED(T))| = {}^nc_2$. Then since T has no source, for any $u \in V(T)$, all the in-neighbors v of u are eccentric vertices of u. This happens only if $d_T(u,v) = 2$ for all $v \in N_T^-(u)$ or $d_T(u,v) = \infty$ for all $u \in N_T^-(u)$. □

Theorem 2.6 *For any graph G, $ED(G)$ is not a tournament.*

Proof. Suppose $ED(G)$ is a tournament. Let u and v be any two adjacent vertices in G. Since $ED(G)$ is a tournament, either v is an eccentric vertex of u or u is an eccentric vertex of v.

Without loss of generality, let v be an eccentric vertex of u. Then $e(u) = 1$ and hence all the vertices of $G-u$ are adjacent as well as eccentric vertices of u in G. It follows that $diam(G) = 2$. Hence for two non-adjacent vertices x and y in G, x is an eccentric vertex of y and y is an eccentric vertex of x. Hence (x,y) and (y,x) are arcs in $ED(G)$, which is a contradiction. Hence $ED(G)$ is not a tournament. □

Theorem 2.7. *Let $T=(V,A)$ be a tournament on n vertices. If T has a source, then $|A(ED(T))| \leq \binom{n}{2} + n - 1$ and equality holds if and only if T is a transitive tournament.*

Proof. Let w be the source of T. Then $d^+(w)=s_T(w)=n-1$ and all the vertices in $T-\{w\}$ are eccentric vertices of w. Hence $|E(w)|=n-1$. Also for any vertex $u\neq w$, only in-neighbors of u can be eccentric vertices of u and hence $|E(u)|\leq |I(u)|=d^-(u)$.

$$\text{Thus } |A(ED(T))| = \sum_{u\in V} |E(u)|$$
$$= |E(w)| + \sum_{u\in V-\{w\}} |E(u)|$$
$$\leq \binom{n}{2} + n - 1.$$

Now, suppose T is a transitive tournament. Then T has a Hamilton path $(u_1,u_2,\ldots,u_n)$. Since T is transitive it follows that $(u_i,u_j)\in A(T)$ for all i,j with $i<j$. Hence $|E(u_i)|=d^-(u_i)=i-1$.

$$\text{Thus } |A(ED(T))| = \sum_{i=1}^{n} |E(u_i)|$$
$$= \sum_{i=2}^{n} d^-(u_i) + (n-1)$$
$$= \binom{n}{2} + n - 1.$$

Conversely, suppose $|A(ED(T))| = \binom{n}{2} + (n-1)$. Then for any $u\in V-\{w\}$ all the in-neighbors of u are the only eccentric vertices of u. Further every vertex of $V-\{w\}$ is an eccentric vertex of w. Hence $\sum_{u\in V-\{w\}} d^-(u) = \binom{n}{2}$ and $1\leq d^-(u)\leq n-1$ for all $u\in V-\{w\}$. We claim that no two vertices of T have the same score. Let $u,v\in V(T)$ and assume without loss of generality that $(u,v)\in A(T)$. Let $W=O(v)$ so that $s_T(v)=|W|$. Now since $(v,w)\in A(T)$ for all $w\in W$, $(u,v)\in A(T)$ and T is transitive it follows that $(u,w)\in A(T)$. Thus $d^+(u)\geq 1+|W|>d^+(v)$. Hence the score sequence of T is $0,1,2,\ldots,n-1$ and it follows from Theorem 1.1 that T is a transitive tournament. □

3 Conclusion and Scope

In this paper we have presented several basic results on eccentric digraphs of tournaments. Miller et al. [8] have presented several open problems and conjectures on eccentric digraphs. In particular one can investigate corresponding problems for eccentric digraphs of tournaments.

ACKNOWLEDGEMENTS.
This research work is supported by RUI grant # 1001/PKOMP/811290 awarded by Universiti Sains Malaysia.

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