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Decomposition of bipartite graphs arising in cellular manufacturing systems

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Abstract

The input for a cellular manufacturing problem consists of a set X of m machines, a set Y of p parts and an $m \times p$ matrix $A = (a_{ij})$, where $a_{ij} = 1$ or 0 according as the part p_j is processed on the machine m_i. This data can be represented as a bipartite graph with bipartition X, Y where m_i is joined to p_j if $a_{ij} = 1$. Let $\pi = \{G_1, G_2, \ldots, G_k\}$ be a set of connected subgraphs of G, such that $\{V(G_1), V(G_2), \ldots, V(G_k)\}$ forms a partition of $V(G)$. Let $\beta(G,\pi)$ denote the total number of edges in G with one end in $V(G_i)$ and other end in $V(G_j)$ for all i and j with $i \neq j$. Let $\beta(G,k) = \min_{\pi} \beta(G,\pi)$ where the minimum is taken over all partitions π of G into a set of k connected subgraphs. In this paper we obtain bounds for $\beta(G,k)$ and determine its value for several classes of graphs.

Keywords: *Bipartite Graph, Exceptional Element, Cellular Manufacturing System, Part Grouping.*

1 Introduction

Cellular manufacturing is an application of the principles of group technology in manufacturing. The input for a cellular manufacturing problem consists of a set X of m machines, a set Y of p parts and an $m \times p$ matrix $A = (a_{ij})$, where $a_{ij} = 1$ or 0 according as the part p_j is processed on the machine m_i. This data can be represented as a bipartite graph with bipartition X, Y where m_i is joined to p_j if $a_{ij} = 1$. Those parts which require a similar manufacturing process are grouped into a family, called a part family. Given a part family, a group of machines is identified for manufacturing the parts of the family and the part family along with the corresponding group of machines is called a cell. Thus a cell is a small scale, well-defined production unit within a large factory, which has the responsibility for producing a family of parts. Cellular manufacturing problem is to design cells in such a way that some measure of performance is optimized. We confine ourselves to the problem of minimizing the number of part movements from one cell to another cell. Cell formation problem in cellular manufacturing system is a NP-hard problem [7]. Many authors have proposed several approaches for this problem such as mathematical programming [1], neural network [3], graph theoretic approach [6], genetic algorithm [8], Boolean matrix approach [9], and clustering approach [10].

The machine-part incidence matrix $A = (a_{ij})$ of a cellular manufacturing problem can be represented as a bipartite graph $G = (V,E)$ where $V = X \cup Y$ and a machine $m_i \in X$ is joined to part $p_j \in Y$ if $a_{ij} = 1$. Let $\pi = \{G_1, G_2, \ldots, G_k\}$ be a set of connected subgraphs of G, such that $\{V(G_1), V(G_2), \ldots, V(G_k)\}$ forms a partition of $V(G)$. Let $\beta(G,\pi)$ denote the total number of edges in G with one end in $V(G_i)$ and other end in $V(G_j)$ for all i and j with $i \neq j$. An edge of G with one end in G_i and other end in G_j is called an exceptional edge with respect to the partition π. Let $\beta(G,k) = \min_\pi \beta(G,\pi)$ where the minimum is taken over all partitions π of G into a set of k connected subgraphs. If $G_1, G_2, \ldots, G_k$ are taken as cells, then $\beta(G,k)$ gives the minimum number of exceptional edges for the given cellular manufacturing problem.

In this paper we obtain bounds for $\beta(G,k)$ and determine its value for several classes of graphs. For graph theoretic terminology we refer to Chartrand and Lesniak [2].

2 Main Results

Let $G = (V,E)$ be a bipartite graph with k- non-trivial components $G_1, G_2, \ldots, G_k$. Then $V(G_1), V(G_2), \ldots, V(G_k)$ can be taken as cells and $\beta(G,k) = 0$. Conversely if $\beta(G,k) = 0$, then $V(G)$ can be partitioned into connected subgraphs $G_1, G_2, \ldots, G_k$ such that $\{V(G_1), V(G_2), \ldots, V(G_k)\}$ forms a partition of $V(G)$ and there is no edge with one end in $V(G_i)$ and other end in $V(G_j)$ for $i \neq j$. Thus G is a disconnected graph with k-components. Hence $\beta(G,k) = 0$ if and only if G is a disconnected graph with k-components.

Example 1. Consider the following problem of cell design given in [4] in which the machine-part incidence matrix is given below.

$$A = \begin{array}{c} \\ m_1 \\ m_2 \\ m_3 \\ m_4 \end{array} \begin{array}{c} \begin{array}{ccccc} p_1 & p_2 & p_3 & p_4 & p_5 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

The bipartite graph G represented by A is given in Figure 1. Since G is a disconnected graph with two components, $\beta(G,2) = 0$ and the two cells are given by $V(G_1) = \{m_1, m_3, p_2, p_4, p_5\}$ and $V(G_2) = \{m_2, m_4, p_1, p_3\}$, This solution is the same as the one given in [4].

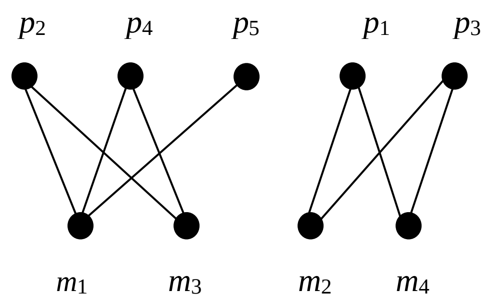

Fig. 1

Example 2. Consider the following problem of cell design given in [5] in which the machine-part incidence matrix is given by

$$B = \begin{array}{c} \\ m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \\ m_7 \\ m_8 \\ m_9 \\ m_{10} \end{array} \begin{array}{c} \begin{array}{cccccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

The bipartite graph G represented by B is given in Figure 2. Since G is a disconnected graph with three components, $\beta(G,3) = 0$ and the three cells are given by $V(G_1) = \{m_1, m_7, m_{10}, p_1, p_4, p_5, p_6\}$, $V(G_2) = \{m_2, m_3, m_4, m_8, p_2, p_7, p_9, p_{10}\}$ and $V(G_3) = \{m_5, m_6, m_9, p_3, p_8\}$. This solution is the same as the one given in [5].

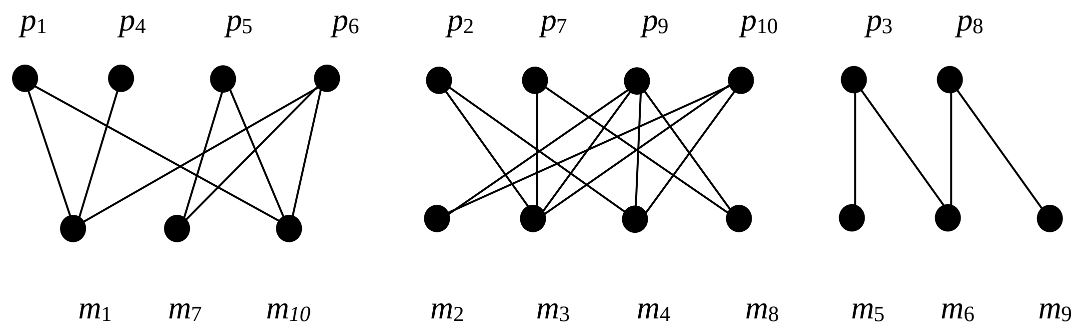

Fig. 2

In the following theorem we determine $\beta(G,k)$ for the complete bipartite graph $K_{m,n}$.

Theorem 1. *For the complete bipartite graph* $G = K_{m,n}$, *we have* $\beta(G,k) = (k-1)(m+n-k)$, *where* $2 \le k \le min\{m, n\}$.

Proof. Let $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$ be the bipartition of G. Let $\pi = \{G_1, G_2, \dots, G_k\}$ be a k-cell partition of $V(G)$, where $V(G_i) = \{a_i, b_i\}$, $1 \le i \le k - 1$ and $V(G_k) = \{a_k, a_{k+1}, \dots, a_m, b_k, b_{k+1}, \dots, b_n\}$. Then the number of exceptional edges with respect to the partition π is equal to $(k-1)(m + n - k)$. Hence $\beta(G,k) \le (k-1)(m+n-k)$. Now, let $\pi' = \{G_1', G_2', \dots, G_k'\}$ be any arbitrary k-cell partition of $V(G)$. Let $V(G'_i) = A_i \cup B_i$, $A_i \subseteq A$ and $B_i \subseteq B$, $|A_i| = m_i \ge 1$ and $|B_i| = n_i \ge 1$ and $\bigcup_{i=1}^{k} A_i = A$, $\bigcup_{i=1}^{k} B_i = B$. Since $m_i n_j \ge m_i + n_j - 1$ for $1 \le i, j \le k$, it follows that

$$\begin{aligned}\beta(G,\pi') &= m_1(n_2 + n_3 + \dots + n_k) + m_2(n_1 + n_3 + \dots + n_k) + \dots + \\ &\quad m_k(n_1 + n_2 + \dots + n_{k-1}) \\ &\ge (k-1)(m_1 + m_2 + \dots + m_k) + (k-1)(n_1 + n_2 + \dots + n_k) - k(k-1) \\ &= (k-1)(m+n) - k(k-1) \\ &= (k-1)(m+n-k).\end{aligned}$$

Hence $\beta(G,k) \ge (k-1)(m+n-k)$. Thus $\beta(G,k) = (k-1)(m+n-k)$.

In the following theorem we give a lower bound for the parameter $\beta(G,k)$ and characterize all graphs that attain the bound. □

Theorem 2. *For any connected bipartite graph G, we have* $\beta(G,k) \geq (k-1)$. *Further* $\beta(G,k) = (k-1)$ *if and only if G can be obtained from a tree T of order k with* $V(T) = \{v_1, v_2, \ldots, v_k\}$ *and k non-trivial connected bipartite graphs* $G_1, G_2, \ldots, G_k$ *by identifying a vertex of* G_i *with* v_i, $1 \leq i \leq k$.

Proof. Let $\pi = \{V(G_1), V(G_2), \ldots, V(G_k)\}$ be a k-cell partition of $V(G)$ such that $\langle V(G_i)\rangle$ is connected. We define a graph H with $V(H) = \{G_1, G_2, \ldots, G_k\}$ and G_i is joined to G_j if $i \neq j$ and there exists an edge with one end in G_i and the other end in G_j. Since G is connected, it follows that H is also connected. Hence $|E(H)| \geq k - 1$. Further any edge in H gives at least one exceptional edge for π and hence $\beta(G,k) \geq (k-1)$. Now $\beta(G,k) = (k-1)$ if and only if $|E(H)| = k - 1$, or equivalently H is a tree, and every edge in H gives rise to exactly one exceptional edge for π. Hence the result follows. □

Corollary 1. *For any tree T, we have* $\beta(G,k) = (k-1)$.

Corollary 2. *For any even cycle* C_n*,we have* $\beta(C_n,k) = k$.

Proof. Let $\pi = \{G_1, G_2, \ldots, G_k\}$ be a partition of $V(C_n)$ into k-subsets such $\langle V(G_i)\rangle$ is a nontrivial connected graph. Then $\langle V(G_i)\rangle$ is a path and the graph H is a cycle on k vertices. Further each of the k-edges of H gives rise to exactly one exceptional edge for π. Thus for every partition π of C_n into k subsets, the number of exceptional edges is k and hence $\beta(C_n,k) = k$. □

Theorem 3. Let $G = K_{n,n} - M$, where M is a perfect matching in $K_{n,n}$ and let $2 \leq k \leq n$. Then $\beta(G,k) = k(2n+1-k) - 3n + \max\{n-2k+2, 0\}$.

Proof. Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be the bipartition of $V(G)$ and let $M = \{a_ib_i : i = 1, \ldots, n\}$. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a k-cell partition of $V(G)$, where $V_i = A_i \cup B_i$, $A_i = \{a_i\}$, for $1 \leq i \leq k-1$, $A_k = \{a_k, a_{k+1}, \ldots, a_n\}$,

$$B_i = \begin{cases} b_{n-i+1} & \text{if } 1 \leq i \leq \left\lceil \frac{n+1}{2} \right\rceil - 1 \\ b_{i-\left\lceil \frac{n}{2} \right\rceil} & \text{if } \left\lceil \frac{n+1}{2} \right\rceil \leq i \leq k-1, \end{cases}$$

And $B_k = B - \bigcup_{i=1}^{k} B_i$. Let $G_i = \langle A_i \cup B_i \rangle$. Then $|E(\langle G_i \rangle)| = 1$ for $1 \leq i \leq k-1$, and

$$G_k = \begin{cases} K_{n-k+1,n-k+1} - S_1 & \text{if } n \geq 2k-2, S_1 \subset M, |S_1| = n-2k+2 \\ K_{n-k+1,n-k+1} & \text{if } n \leq 2k-2. \end{cases}$$

Hence it follows that

$$\beta(G,\pi)=\begin{cases} k(2n-k+1)-3n+(n-2k+2) & if \quad n>2k-2, \\ k(2n-k+1)-3n & if \; n\le 2k-2. \end{cases}$$

Thus $\beta(G,k)\le k(2n+1-k)-3n+\max\{n-2k+2,0\}$.

Now let π = $\{V_1, V_2, \ldots, V_k\}$ be any arbitrary k-cell partition of $V(G)$, $V_i = A_i \cup B_i$, $1 \le i \le k$. Without loss of generality, assume that $|A_i| \le |A_{i+1}|$, $1 \le i \le k-1$. If $|A_j| < |A_{j+1}|$, then a vertex u can be transferred from B_j to B_{j+1} without increasing the value of β, since the vertex u has exactly one non-neighbor in A. We repeat this process of transferring vertices until $|B_j| < |B_{j+1}|$. Thus we may assume that $|B_j| < |B_{j+1}|$ whenever $|A_j| < |A_{j+1}|$. Suppose there exists j such that $|A_j| = |A_{j+1}| \ge 2$, and $|B_j| = |B_{j+1}| \ge 2$. Then choose $c \in A_j$ and $d \in B_j$. Consider the new partition π' = $\{C_1 \cup D_1, C_2 \cup D_2, \ldots, C_k \cup D_k\}$, where $C_j = A_j - \{c\}$, $D_j = B_j - \{d\}$, $C_{j+1} = A_j \cup \{c\}$, $D_{j+1} = B_j \cup \{d\}$, and $C_i = A_i$, $D_i = B_i$ for $i \ne j, j+1$.

Let G_j denote the subgraph of G induced by $\langle C_j \cup D_j \rangle$, $1 \le j \le k$. We claim that $\beta(G,\pi')=\beta(G,\pi)$. Let x denote the unique non-neighbor of c in B and let y denote the unique non-neighbor of d in A. If c and d are non-adjacent, then $d = x$, $c = y$ and in this case $\beta(G,\pi')=\beta(G,\pi)$. Now, suppose c and d are adjacent. If $y \in B_j \cup B_{j+1}$ or if $x \in A_j \cup A_{j+1}$ then $\beta(G,\pi')=\beta(G,\pi)-1$. In all the other cases, $\beta(G,\pi')=\beta(G,\pi)$. Thus $\beta(G,\pi')\le\beta(G,\pi)$. Continuing this type of transferring of vertices, we get a partition π'' = $\{A_1'' \cup B_1'', A_2'' \cup B_2'', \ldots, A_k'' \cup B_k''\}$ of $V(G)$, such that $|A_i''| = |B_i''| = 1$, $1 \le i \le k-1$ and $|A_k''| = |B_k''| = n-k+1$.

Let $G_i'' = \langle A_i'' \cup B_i'' \rangle$, $1 \le i \le k$. Any perfect matching M of of $K_{n,n}$ can match at most $k-1$ vertices of A_k'' to the vertices of $B_1'' \cup B_2'' \cup \ldots \cup B_{k-1}''$ and at most $k-1$ vertices of B_k'' to the vertices of $A_1'' \cup A_2'' \cup \ldots \cup A_{k-1}''$. Hence,

$$\begin{aligned} \left|E\left(\langle G_k'' \rangle\right)\right| &\ge (n-k+1)^2 - (n-2k+2) \\ &\ge (n-k+1)^2 - \max\{(n-2k+2), 0\}. \end{aligned}$$

Thus

$$\begin{aligned} \beta(G,\pi'') &\ge (n^2-n) - [(k-1) + (n-k+1)^2 - \max\{(n-2k+2), 0\}] \\ &= k(2n+1-k) - 3n + \max\{(n-2k+2), 0\}. \end{aligned}$$

Hence $\beta(G,k)\ge k(2n+1-k)-3n+\max\{n-2k+2,0\}$.

Thus $\beta(G,k)=k(2n+1-k)-3n+\max\{n-2k+2,0\}$. □

3 Concluding Remarks

In this paper we have investigated the cellular manufacturing design using graph theoretic approach and we have determined the exact value of the number of exceptional edges for some classes of bipartite graphs. An algorithmic study of this problem using standard graph algorithms will be reported in our future paper.

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